Pricing Tranches of a CDO and a CDS Index: Recent Advances and Future Research

Dezhong Wang, Svetlozar T. Rachev, Frank J. Fabozzi

This Version: October, 2006

Dezhong Wang
Department of Applied Probability and Statistics,
University of California, Santa Barbara,
CA 93106-3110, USA
E-mail: dwang@pstat.ucsb.edu

Svetlozar T. Rachev
Chair-Professor, Chair of Econometrics,
Statistics and Mathematical Finance School of
Economics and Business Engineering, University of Karlsruhe,
Postfach 6980, 76128 Karlsruhe, Germany
and
Department of Statistics and Applied Probability,
University of California, Santa Barbara,
CA93106-3110, USA
E-mail: rachev@statistik.uni-karlsruhe.de

Frank J. Fabozzi
Professor in the Practice of Finance,
Yale School of Management,
135 Prospect Street, Box 208200,
New Haven, Connecticut 06520-8200, USA
E-mail: frank-fabozzi@yale.edu

Abstract

In this paper, we review recent advances in pricing tranches of a collateralized debt obligations and credit default swap indexes: one factor Gaussian copula model and its extensions, the structural model, and the loss process model. Then, we propose using heavy-tailed functions in future research. As background, we provide a brief explanation of collateralized debt obligations, credit default swaps, and index tranches.

Keywords and Phrases: *Collateralized Debt Obligation, Credit Default Swap, Credit Default Swap Index, Credit Default Swap Index Tranches.*

1 Introduction

In the recent years, the market for credit derivatives has developed rapidly with the introduction of new contracts and the standardization documentation. These include credit default swaps, basket default swaps, credit default swap indexes, collateralized debt obligations, and credit default swap index tranches. Along with the introduction of new products comes the issue of how to price them. For single-name credit default swaps, there are several factor models (one-factor and two-factor models) proposed in the literature. However, for credit portfolios, much work has to be done in formulating models that fit market data. The difficulty in modeling lies in estimating the correlation risk for a portfolio of credits. In an April 16, 2004 article in the *Financial Times* (Duffie (2004)), Darrell Duffie made the following comment on modeling portfolio credit risk: "Banks, insurance companies and other financial institutions managing portfolios of credit risk need an integrated model, one that reflects correlations in default and changes in market spreads. Yet no such model exists." Almost a year later, a March 2005 publication by the Bank for

International Settlements noted that while a few models have been proposed, the modeling of these correlations is "complex and not yet fully developed." (Amato and Gyntelberg (2005)).

In this paper, first we review three methodologies for pricing CDO tranches. They are the one-factor copula model, the structural model, and the loss process model. Then we propose how the models can be improved.

The paper is structured as follows. In the next section we review credit default swaps and in Section 3 we review collateralized debt obligations and credit default swap index tranches. The three pricing models are reviewed in Sections 4 (one-factor copula model), 5 (structural model), and 6 (loss process model). Our proposed models are provided in Section 7 and a summary is provided in the final section, Section 8.

2 Overview of Credit Default Swaps

The major risk-transferring instrument developed in the past few years has been the credit default swap. This derivative contract permits market participants to transfer credit risk for individual credits and credit portfolios. Credit default swaps are classified as follows: single-name swaps, basket swaps, and credit default index swaps.

2.1 Single-Name Credit Default Swap

A single-name credit default swap (CDS) involves two parties: a protection seller and a protection buyer. The protection buyer pays the protection seller a swap premium on a specified amount of face value of bonds (the notional principal) from an individual company (reference entity/reference credit). In return the protection seller pays the protection buyer an amount to compensate for the loss of the protection buyer upon the occurrence of a credit event with respect to the underlying reference entity.

In the documentation of a CDS contract, a credit event is defined. The list of credit events in a CDS contract may include one or more of the following: bankruptcy or insolvency of the reference entity, failure to pay an amount above a specified threshold over a specified period, and financial or debt restructuring. The swap premium is paid on a series of dates, usually quarterly in arrears based on the actual/360 date count convention.

In the absence of a credit event, the protection buyer will make a quarterly swap premium payment until the expiration of a CDS contract. If a credit event occurs, two things happen. First, the protection buyer pays the accrued premium from the last payment date to the time of the credit event to the seller (on a days fraction basis). After that payment, there are no further payments of the swap premium by the protection buyer to the protection seller. Second, the protection seller makes a payment to the protection buyer. There can be either cash settlement or physical settlement. In cash settlement, the protection seller pays the protection buyer an amount of cash equal to the difference between the notional principal and the present value of an amount of bonds, whose face value equals the notional principal, after a credit event. In physical settlement, the protection seller pays the protection buyer the notional principal, and the protection buyer delivers to the protection seller bonds whose face value equals the notional principal. At the time of this writing, the market practice is physical settlement.

2.2 Basket Default Swap

A basket default swap is a credit derivative on a portfolio of reference entities. The simplest basket default swaps are first-to-default swaps, second-to-default swaps, and nth-to-default swaps. With respect to a basket of reference entities, a first-to-default swap provides insurance for only the first default, a second-to-default swap provides insurance for only the second default, an nth-to-default swap provides insurance for only the nth default. For example, in an nth-to-default swap, the protection seller does not make a payment to the protection buyer for the first $n-1$ defaulted reference entities, and makes a payment for the nth defaulted reference entity. Once there is a payment upopn the default of the nth defaulted reference entity, the swap terminates. Unlike a single-name CDS, the preferred settlement method for a basket default swap is cash settlement.

2.3 Credit Default Swap Index

A credit default swap index (denoted by CDX) contract provides protection against the credit risk of a standardized basket of reference entities. The mechanics of a CDX are slightly different from that of a single-name CDS. If a credit event occurs, the swap premium payment ceases in the case of a single-name CDS. In contrast, for a CDX the swap premium payment continues to be made by the protection buyer but based on a reduced notional amount since less reference entities are being protected. As of this writing, the settlement for a CDX is physical settlement.[1]

Currently, there are two families of standardized indexes: the Dow Jones CDX[2]

[1] The market is considering moving to cash settlement because of the cost of delivering an odd lot in the case of a credit event for a reference entity. For example, if the notional amount of a contract is $20 million and a credit event occurs, the protection buyer would have to deliver to the protection seller bonds of the reference entity with a face value of $160,000. Neither the protection buyer nor the protection seller likes to deal with such a small position.

[2] www.djindexes.com/mdsidx/?index=cdx.

and the International Index Company iTraxx.[3] The former includes reference entities in North America and emerging markets, while the latter includes reference entities in Europe and Asia markets. Both families of indexes are standardized in terms of the index composition procedure, premium payment, and maturity.

The two most actively traded indexes are the Dow Jones CDX NA IG index and the iTraxx Europe index. The former includes 125 North American investment-grade companies. The latter includes 125 European investment-grade companies. For both indexes, each company is equally weighted. Also for these two indexes, CDX contracts with 3-, 5-, 7- and 10-year maturities are available.

The composition of reference entities included in a CDX are renewed every six months based on the vote of participating dealers. The start date of a new version index is referred to as the *roll date*. The roll date is March 20 and September 20 of a calender year or the following business days if these days are not business days. A new version index will be "on-the-run" for the next six months. The composition of each version of a CDX remains static in its lifetime if no default occurs to the underlying reference entities, and the defaulted reference entities are eliminated from the index.

There are two kinds of contracts on CDXs: unfunded and funded. An unfunded contract is a CDS on a portfolio of names. This kind of contract is traded on all the Dow Jones CDX and the iTraxx indexes. For some CDXs such as the Dow Jones CDX NA HY index and its sub-indexes[4] and the iTraxx Europe index, the funded contract is traded. A funded contract is a credit-linked note (CLN), allowing investors who because of client imposed or regulatory restrictions are not permitted to invest in derivatives to gain risk exposure to the CDX market. The funded

[3] www.indexco.com.

[4] The Dow Jones CDX NA HY index includes 100 equal-weighted North America High Yield reference entities. Its sub-indexes include the CDX NA HY B (B-rated), CDX NA HY BB (BB-rated), and CDX NA HY HB (High Beta) indexes.

contract works like a corporate bond with some slight differences. A corporate bond ceases when a default occurs to the reference entity. If a default occurs to a reference entity in an index, the reference entity is removed from the index (and also from the funded contract). The funded contract continues with a reduced notional principal for the surviving reference entities in the index. Unlike the unfunded contract which uses physical settlement, the settlement method for the funded contract is cash settlement.

The index swap premium of a new version index is determined before the roll day and unchanged over its life time, which is referred to as the *coupon* or the *deal spread*. The price difference between the prevailing market spread and the deal spread is paid upfront. If the prevailing market spread is higher than the deal spread, the protection buyer pays the price difference to the protection seller. If the prevailing market spread is less than the deal spread, the protection seller pays the price difference to the protection buyer. The index premium payments are standardized quarterly in arrears on the 20th of March, June, September, and December of each calendar year.

The CDXs have many attractive properties for investors. Compared with the single-name swaps, the CDXs have the advantages of diversification and efficiency. Compared with basket default swaps and collateralized debt obligations, the CDXs have the advantages of standardization and transparency. The CDXs are traded more actively than the single-name CDSs, with low bid-ask spreads.

3 CDOs and CDS Index Tranches

Based on the technology of basket default swaps, the layer protection technology is developed for protecting portfolio credit risk. Basket default swaps provide the protection to a single default in a portfolio of reference entities, for example, the first default, the second default, and the nth default. Correspondingly, there are the first layer protection, the second layer protection, and the nth layer protection. These protection layers work like basket default swaps with some differences. The main difference is that the n basket default swap protects the nth default in a portfolio and the nth protection layer protects the nth layer of the principal of a portfolio, which is specified by a range of percentage, for example 15-20%. The layer protection derivative products include collateralized debt obligations and CDS index tranches.

3.1 Collateralized Debt Obligation

A collateralized debt obligation (CDO) is a security backed by a diversified pool of one or more kinds of debt obligations such as bonds, loans, credit default swaps or structured products (mortgage-backed securities, asset-backed securities, and even other CDOs). A CDO can be initiated by one or more of the following: banks, nonbank financial institutions, and asset management companies, is referred to as the *sponsor*. The sponsor of a CDO creates a company so-called the *special purpose vehicle* (SPV). The SPV works as an independent entity. In this way, CDO investors are isolated from the credit risk of the sponsor. Moreover, the SPV is responsible for the administration. The SPV obtains the credit risk exposure by purchasing debt obligations (bonds or residential and commercial loans) or selling CDSs; it transfers the credit risk by issuing debt obligations (tranches/credit-linked notes). The investors in the tranches of a CDO have the ultimate credit risk exposure to the underlying reference entities.

Figure 1 shows the basic structure of a CDO backed by a portfolio of bonds. The SPV issues four kinds of CLNs referred to as *tranches*. Each tranche has an attachment percentage and a detachment percentage. When the cumulative

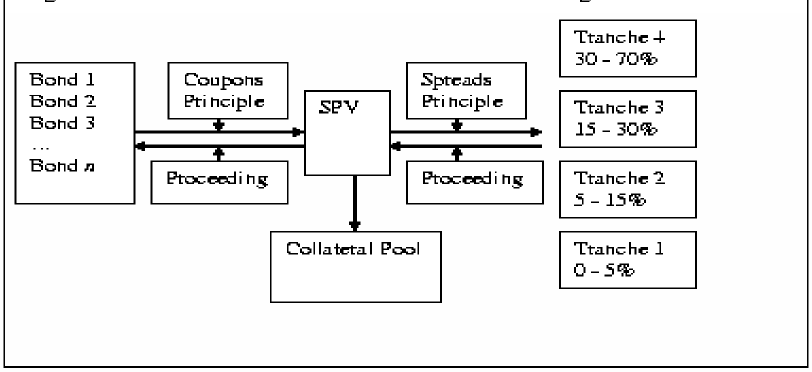

Figure 1: Structure of a Collateralized Debt Obligation

percentage loss of the portfolio of bonds reaches the attachment percentage, investors in the tranche start to lose their principal, and when the cumulative percentage loss of principal reaches the detachment percentage, the investors in the tranche lose all their principal and no further loss can occur to them. For example, in Figure 1 the second tranche has an attachment percentage of 5% and a detachment percentage of 15%. The tranche will be used to covered the cumulative loss during the life of a CDO in excess of 5% (its attachment percentage) and up to a maximum of 15% (its detachment percentage).

In the literature, tranches of a CDO are classified as subordinate/equity tranche, mezzanine tranches, and senior tranches according to their subordinate levels.[5] For example, in Figure 1 tranche 1 is an equity tranche, tranches 2 and 3 are mezzanine tranches, and tranche 4 is a senior tranche. Because the equity tranche is extremely risky, the sponsor of a CDO holds the equity tranche and the SPV sells other tranches to investors.

If the SPV of a CDO actually owns the underlying debt obligations, the CDO is referred to as a *cash CDO*. Cash CDOs can be classified as collateralized bond

[5]See Lucas, Goodman, and Fabozzi (2006).

obligations (CBO) and collateralized loan obligation (CLO). The former have only bonds in their pool of debt obligations, and the latter have only commercial loans in their pool of debt obligations. If the SPV of a CDO does not own the debt obligations, instead obtaining the credit risk exposure by selling CDSs on the debt obligations of reference entities, the CDO is referred to as a *synthetic CDO*.

Based on the motivation of sponsors, CDOs can be classified as *balance sheet* CDOs and *arbitrage* CDOs. The motivation of balance sheet CDOs (primarily CLO) is to transfer the risk of loans in a sponsoring bank's portfolio in order to reduce regulatory capital requirements. The motivation of arbitrage CDOs is to arbitrage the price difference between the underlying pool of debt obligations and CDO tranches.

3.2 CDS index tranches

With the innovation of CDXs, the synthetic CDO technology is applied to slice CDXs into standardized tranches with different subordinate levels to satisfy investors with different risk favorites. The tranches of an index provide the layer protections to the underlying portfolio risk in the same way as the tranches of a CDO as has been explained earlier.

Both of the most actively traded indexes— the Dow Jones CDX NA IG and the iTraxx Europe— are sliced into five tranches: equity tranche, junior mezzanine tranche, senior mezzanine tranche, junior senior tranche, and super senior tranche. The standard tranche structure of the Dow Jones CDX NA IG is 0-3%, 3-7%, 7-10%, 10-15%, and 15-30%. The standard tranche structure of the iTraxx Europe is 0-3%, 3-6%, 6-9%, 9-12%, and 12-22%.

Table 1 shows the index and tranches market quotes for the CDX NA IG and the iTraxx Europe on August 4, 2004. For both indexes, the swap premium of the

Table 1: CDS Index and Tranche Market Quotes—August 4, 2004

iTraxx Europe (5 year)					
index	0-3%	3-6%	6-9%	9-12%	12-22%
42	27.6%	168	70	43	20

CDX NA IG (5 year)					
index	0-3%	3-7%	7-10%	10-15%	15-30%
63.25	48.1%	347	135.5	47.5	14.5

Data are collected by GFI Group Inc. and used in Hull and White (2004)

equity tranche is paid differently from the non-equity tranches. It includes two parts: (1) the upfront percentage payment and (2) the fixed 500 basis points premium per annual. The market quote is the upfront percentage payment. For example, the market quote of 27.8% for the iTraxx equity tranche means that the protection buyer pays the protection seller 27.8% of the principal upfront. In addition to the upfront payment, the protection buyer also pays the protection seller the fixed 500 basis points premium per annual on the outstanding principal. For all the non-equity tranches, the market quotes are the premium in basis points, paid quarterly in arrears. Just like the indexes, the premium payments for the tranches (with the exception of the upfront percentage payment of the equity tranche) are made on the 20th of March, June, September, and December of each calendar year.

Following the commonly accepted definition for a synthetic CDO, CDX tranches are not part of a synthetic CDO because they are not backed by a portfolio of bonds or CDSs (Hull and White (2004)). In addition, CDX tranches are unfunded and they are insurance contracts, while synthetic CDO tranches are funded and they are CLNs. However, the net cash flows of index tranches are the same as synthetic CDO tranches and these tranches can be priced the same way as a synthetic CDO.

4 One-Factor Copula Model

The critical input into pricing a synthetic CDO and CDS index tranches is an estimate of the default dependence (default correlation) between the underlying assets. One popular method for estimating the dependence structure is using copula functions, a method first applied in actuarial science. While there are several types of copula function models, Li (1999, 2000) introduces the one-factor Gaussian copula model for the case of two companies and Laurent and Gregory (2003) extend the model to the case of N companies. Several extensions to the one-factor Gaussian copula model are subsequently introduced into the literature. In this section, we provide a general description of the one-factor copula function, introduce the market standard model, and review both the one-factor double t copula model (Hull and White (2004)) and the one-factor normal inversion Gaussian copula model (Kalemanova, Schmid, and Werner (2005)).

Suppose that a CDO includes n assets $i = 1, 2, \ldots, n$ and the default times τ_i of the ith asset follows a Poisson process with a parameter λ_i. The λ_i is the default intensity of the ith asset. Then the probability of a default occurring before the time t is

$$P(\tau_i < t) = 1 - \exp(-\lambda_i t). \tag{1}$$

In a one-factor copula model, it is assumed that the default time τ_i for the ith company is related to a random variable X_i with a zero mean and a unit variance. For any given time t, there is a corresponding value x such that

$$P(X_i < x) = P(\tau_i < t), \qquad i = 1, 2, \ldots, n. \tag{2}$$

Moreover, the one-factor copula model assumes that each random variable X_i is the sum of two components

$$X_i = a_i M + \sqrt{1 - a_i^2} Z_i, \qquad i = 1, 2, \ldots, n, \tag{3}$$

where Z_i is the idiosyncratic component of company i, and M is the common component of the market. It is assumed that the M and Z_i's are mutually independent random variables. For simplicity, it is also assumed that the random variables M and Z_i's are identical. The factor a_i satisfies $-1 \leq a_i \leq 1$. The default correlation between X_i and X_j is $a_i a_j, (i \neq j)$.

Let F denote the cumulative distribution of the Z_i's and G denote the cumulative distribution of the X_i's. Then given the market condition $M = m$, we have

$$P(Z_i < x | M = m) = F(\frac{x - a_i m}{\sqrt{1 - a_i^2}}), \tag{4}$$

and the conditional default probability is

$$P(\tau_i < t | M = m) = F\{\frac{G^{-1}[P(\tau_i < t)] - a_i m}{\sqrt{1 - a_i^2}}\}. \tag{5}$$

For simplicity, the following two assumptions are made:

- All the companies have the same default intensity, i.e, $\lambda_i = \lambda$.
- The pairwise default correlations are the same, i.e, in equation (3), $a_i = a$.

The second assumption means that the contribution of the market component is the same for all the companies and the correlation between any two companies is constant, $\beta = a^2$.

Under these assumptions, given the market situation $M = m$, all the companies have the same cumulative risk-neutral default probability $D_{t|m}$. Moreover, for a given value of the market component M, the defaults are mutually independent for all the underlying companies. Letting $N_{t|m}$ be the total defaults that have occurred by time t conditional on the market condition $M = m$, then $N_{t|m}$ follows a binomial distribution $\text{Bin}(n, D_{t|m})$, and

$$P(N_{t|m} = j) = \frac{n!}{j!(n-j)!} D_{t|m}^{j} (1 - D_{t|m})^{n-j}, \quad j = 0, 1, 2, \ldots, n. \tag{6}$$

The probability that there will be exactly j defaults by time t is

$$P(N_t = j) = E^M P(N_{t|m}) = \int_{-\infty}^{\infty} P(N_{t|m} = j) f_M(m) dm, \tag{7}$$

where $f_M(m)$ is the probability density function (pdf) of the random variable M.

4.1 Market Standard Model

Li (1999, 2000) was the first to suggest that the Gaussian copula can be employed in credit risk modeling to estimate the correlation default. In a one-factor Gaussian copula model, the distributions of the common market component M and the individual component Z_i's in equation (3) are standard normal Gaussian distributions. Because the sum of two independent Gaussian distributions is still a Gaussian distribution, the X_i's in equation (3) have a closed form. It can be verified that the X_i's have a standard normal distribution.

The one-factor copula Gaussian copula model is the market standard model when implemented under the following assumptions:

- a fixed recovery rate of 40%,
- the same CDS spreads for all of the underlying reference entities,
- the same pairwise correlations,

- the same default intensities for all the underlying reference entities.

The market standard model does not appear to fit market data well (see Hull and White (2004) and Kalemanova et al. (2005)). In practice, market practitioners use *implied correlations* and *base correlations*.

The implied correlation for a CDO tranche is the correlation that makes the value of a contract on the CDO tranche zero when pricing the CDO with the market standard model. For a CDO tranche, when inputting its implied correlation into the market standard model, the simulated price of the tranche should be its market price.

McGinty, Beinstein, Ahluwalia, and Watts (2004) introduced base correlations in CDO pricing. To understand base correlations, let's use an example. Recalling the CDX NA IG tranches 0-3%, 3-7%,7-10%, 10-15%, and 15%-30%, and assuming there exists a sequence of equity tranches 0-3%, 0-7%, 0-10%, 0-15%, and 0-30%, the premium payment on an equity tranche is a combination of the premium payment of the CDX NA IG tranches that are included in the corresponding equity tranche. For example, the equity tranche 0-10% includes three CDX NA IG tranches: 0-3%, 3-7%, and 7-10%. The premium payment on the equity tranche 0-10% includes three parts. The part of 0-3% is paid the same way as the CDX NA IG tranche 0-3%, the part of 3-7% is paid the same way as the CDX NA IG tranche 3-7%, and the part of 7-10% is paid the same way as the CDX NA IG tranche 7-10%. Then the definition of base correlation is the correlation input that make the prices of the contracts on these series of equity tranches zero. For example, the base correlation for the CDX NA IG tranche 7-10% is the implied correlation that makes the price of a contract on the equity tranche 0-10% zero.

4.2 One-Factor Double t Copula Model

The natural extension to a one-factor Gaussian copula model is using heavy-tailed distributions. Hull and White (2004) propose a one-factor double t copula model. In the model, the common market component M and the individual components Z_i in equation (3) are assumed to have a normalized Student's t distribution

$$\begin{array}{ll} M = \sqrt{(n_M - 2)/n_M} T_{n_M}, & T_{n_M} \sim \mathcal{T}(n_M) \\ Z_i = \sqrt{(n_i - 2)/n_i} T_{n_i}, & T_{n_i} \sim \mathcal{T}(n_i) \end{array}, \tag{8}$$

where $\mathcal{T}_n$ is a Student's t distribution with degrees of freedom $n = 3, 4, 5, \ldots$.

In the model, the distributions of X_i's do not have a closed form but instead must be calculated numerically.

Hull and White (2004) find that the one-factor double t copula model fits market prices well when using the Student's t distribution with 4 degrees of freedom for M and Z_i's.

4.3 One-Factor Normal Inverse Gaussian Copula Model

Kalemanova, Schmid, and Werner (2005) propose utilizing normal inverse Gaussian distributions in a one-factor copula model. A normal inverse Gaussian distribution is a mixture of normal and inverse Gaussian distributions.

An inverse Gaussian distribution has the following density function

$$f_{IG}(x; \zeta, \eta) = \begin{cases} \frac{\zeta}{\sqrt{2\pi\eta}} x^{-3/2} \exp(-\frac{(\zeta - \eta x)^2}{2\eta x}), & \text{if} \quad x > 0 \\ 0, & \text{if} \quad x \leq 0 \end{cases}, \tag{9}$$

where $\zeta > 0$ and $\eta > 0$ are two parameters. We denote the inverse Gaussian distribution as $\mathcal{IG}(\zeta, \eta)$.

Suppose Y is an inverse Gaussian distribution. A normal Gaussian distribution $X \sim \mathcal{N}(\upsilon, \sigma^2)$ is a *normal inverse Gaussian* (NIG) distribution when its mean υ and variance σ^2 are random variables as given below

$$\begin{array}{l} \upsilon = \mu + \beta Y, \quad \sigma^2 = Y \\ Y \sim \mathcal{IG}(\delta\gamma, \gamma^2) \end{array}, \tag{10}$$

where $\delta > 0$, $0 \leq |\beta| < \alpha$, and $\gamma := \sqrt{\alpha^2 - \beta^2}$. The distribution of the random variable X is denoted by $X \sim (\alpha, \beta, \mu, \delta)$. The density of X is

$$f(x; \alpha, \beta, \mu, \delta) = \frac{\delta\alpha \exp(\delta\gamma + \beta(x - u))}{\pi\sqrt{\delta^2 + (x - \mu)^2}} K(\alpha\sqrt{\delta^2 + (x - \mu)^2}), \tag{11}$$

where $K(.)$ is the modified Bessel function of the third kind as defined below

$$K(\omega) := \frac{1}{2} \int_0^\infty \exp(-\frac{1}{2}\omega(t - t^{-1}))dt. \tag{12}$$

The mean and variance of the NIG distribution X are respectively

$$E(X) = \mu + \frac{\delta\beta}{\gamma}, \qquad Var(X) = \frac{\delta\alpha^2}{\gamma^3}. \tag{13}$$

The family of NIG distributions has two main properties. One is the closure under the scale transition

$$X \sim \mathcal{NIG}(\alpha, \beta, \mu, \delta) \Rightarrow cX \sim \mathcal{NIG}(\frac{\alpha}{c}, \frac{\beta}{c}, c\mu, c\delta). \tag{14}$$

The other is that if two independent NIG random variables X and Y have the same α and β parameters, then the sum of these two variables is still an NIG variable as shown below

$$\begin{aligned} X \sim \mathcal{NIG}(\alpha, \beta, \mu_1, \delta_1), Y \sim \mathcal{NIG}(\alpha, \beta, \mu_2, \delta) \\ \Rightarrow X + Y \sim \mathcal{NIG}(\alpha, \beta, \mu_1 + \mu_2 \delta_1 + \delta_2) \end{aligned} . \tag{15}$$

When using NIG distributions in a one-factor copula model, the model is referred to as a *one-factor normal inverse Gaussian copula model*. The distributions for M and Z_i's in equation (3) are given below

$$\begin{aligned} &M \sim \mathcal{NIG}(\alpha, \beta, -\frac{\alpha\beta}{\sqrt{\alpha^2 - \beta^2}}, \alpha) \\ &Z_i \sim NIG(\frac{\alpha\sqrt{1-a_i^2}}{a_i}, \frac{\beta\sqrt{1-a_i^2}}{a_i}, -\frac{\alpha\beta\sqrt{1-a_i^2}}{a_i\sqrt{\alpha^2 - \beta^2}}, \frac{\alpha\sqrt{1-a_i^2}}{a_i}). \end{aligned} \tag{16}$$

The distributions of X_i's in equation (3) are

$$X_i \sim \mathcal{NIG}(\frac{\alpha}{a_i}, \frac{\beta}{a_i}, -\frac{\alpha\beta\sqrt{1-a_i^2}}{a_i\sqrt{\alpha^2 - \beta^2}}, \frac{\alpha}{a_i}). \tag{17}$$

The selection of the parameters makes the variables X_i's, M, and Z_i's have a zero mean, and a unit variance when $\beta = 0$.

The one-factor normal inverse Gaussian copula model fits market data a little bit better than the one-factor double t copula model. The advantage of the one-factor normal inverse Gaussian copula model is that the X_i's in the model have a closed form. This makes the computing time is reduced significantly, compared with that of the one-factor double t copula model. The former is about five times faster than the latter.

5 Structural Model

Hull, Predescu, and White (2005) propose the structural model to price the default correlation in tranches of a CDO or an index. The idea is based on Merton's model (1974) and its extension by Black and Cox (1976). It is assumed that the value of a company follows a stochastic process, and if the value of the company goes below a minimum value (barrier), the company defaults.

In the model, N different companies are assumed and the value of company i ($1 \leq i \leq N$) at time t is denoted by V_i. The value of the company follows a stochastic process as shown below

$$dV_i = \mu_i V_i dt + \sigma_i V_i dX_i, \tag{18}$$

where μ_i is the expected growth rate of the value of company i, σ_i is the volatility of the value of company i, and $X_i(t)$ is a variable following a continuous-time Gaussian stochastic process (Wiener process). The barrier for company i is denoted by B_i. Whenever the value of company i goes below the barrier B_i, it defaults.

Without the loss of generality, it is assumed that $X_i(0) = 0$. Applying Ito's formula to $\ln V_i$, it is easy to show that

$$X_i(t) = \frac{\ln V_i(t) - \ln V_i(0) - (\mu_i - \sigma_i^2/2)}{\sigma_i}. \tag{19}$$

Corresponding to B_i, there is a barrier B_i^* for the variable X_i as given below

$$B_i^* = \frac{\ln B_i - \ln V_i(0) - (\mu_i - \sigma_i^2/2)t}{\sigma_i}. \tag{20}$$

When X_i falls below B_i^*, company i defaults. Denote

$$\beta_i = \frac{\ln H_i - \ln V_i(0)}{\sigma_i} \qquad \gamma_i = -\frac{\mu_i - \sigma_i^2/2}{\sigma_i}, \tag{21}$$

then $B_i^* = \beta_i + \gamma_i t$.

To model the default correlation, it is assumed that each Wiener process X_i follows a two-component process which includes a common Wiener process M and an idiosyncratic Wiener process Z_i. It is expressed as

$$dX_i(t) = a_i(t)dM(t) + \sqrt{1 - a_i^2(t)}dZ_i(t), \tag{22}$$

where the variable $a_i, 1 \leq a_i \leq 1$ is used to control the weight of the two-component process. The Wiener processes M and Z_i's are uncorrelated with each other. In this model, the default correlation between two companies i and j is $a_i a_j$.

The model can be implemented by Monte Carlo simulation. Hull, Predescu, and White (2005) implement the model in the three different ways:

- *Base case*: constant correlation and constant recovery rate.
- *Stochastic Corr.*: stochastic correlation and constant recovery rate.
- *Stochastic RR*: stochastic correlation and stochastic recovery rate.

Two comparisons between the base-case structural model and the one-factor Gaussian copula model are provided. One is to calculate the joint default probabilities of two companies by both models. The other is to simulate the iTraxx Europe index tranche market quote by both models. In both cases, the results of these models are very close when the same default time correlations are input, while the one-factor Gaussian copula is a good approximation to the base-case structural model, the structural model has two advantages: it is a dynamic model and it has a clear economic rationale.

6 Loss Process Model

Loss process models for pricing correlation risk have been developed by Schönbucher (2005), Sidenius et al. (2005), Di Graziano and Rogers(2005), and Bennani (2005). Here we introduce the basic idea of the loss process model as discussed by Schönbucher. We omit the mathematical details.

6.1 Model Setup

The model is set up in the probability space $(\Omega, (\mathcal{F}_t)_{0\le t\le T}, Q)$, where Q is a spot martingale measure, $(F_t)_{0\le t\le T}$ is the filtration satisfying the common definitions, and Ω is the sample space. Assume that there are N company names in a portfolio. Each name has the same notional principal in the portfolio. Under the assumption of a homogenous recovery rate for all the companies, all companies have identical losses in default which is normalized to one. The cumulative *default loss process* is defined by

$$L_t = \sum_{k}^{N} 1_{\{\tau_k \le t\}}, \tag{23}$$

where τ_k is the default time of company k, and the default indicator $1_{\{\tau_k\le t\}}$ is 1 when $\tau_k \le t$ and 0 when $\tau_k > t$. The loss process is an N-bounded, integer-valued, non-decreasing Markov chain. Under Q-measure, the probability distribution of $L(T)$ at time $t < T$ is denoted by the vector $\boldsymbol{p}(t,T) := (p_0(t,T), \ldots, p_N(t,T))'$, where the p_i's are conditional probabilities

$$p_i(t,T) := P[L(T) = i|\mathcal{F}_t], \qquad i = 0, 2, \ldots, N, t \le T. \tag{24}$$

The conditional probability $p_i(t,T)$ is the implied probability of $L(T) = i, T \geq t$ given the information up to time t. $\boldsymbol{p}(t,.)$ is referred to as the *loss distribution* at time t.

6.2 Static Loss Process

To price a CDO, it is necessary to determine an implied initial loss distribution $\boldsymbol{p}(0,T)$. The implied initial loss distribution can be found by solving the evolution of the loss process $L(t)$. As the loss process $L(t)$ is an inhomogeneous Markov chain in a finite state space with $N+1$ states $\{0,1,2,\ldots,N\}$, its transition probabilities are uniquely determined by its generator matrix.

Assuming that there is only one-step transition at any given time t, the generator matrix of the loss process has the following form

$$A(t) = \begin{pmatrix} -\lambda_0(t) & \lambda_0(t) & 0 & \ldots & 0 & 0 \\ 0 & -\lambda_1(t) & \lambda_1(t) & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -\lambda_{N-1}(t) & \lambda_{N-1}(t) \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \tag{25}$$

where the $\lambda_i(t)'s$ are the transition rates $i = 0,1,\ldots,N-1$. The state N is an absorbing state.

The probability transition matrix, defined by $P_{ij}(t,T) := P[L(T) = j|L(t) = i]$, satisfies the following Kolmogorov equations

$$\begin{aligned} &\frac{d}{dT}P_{i,0}(t,T) = -\lambda_0(T)P_{i,0}(t,T) \\ &\frac{d}{dT}P_{i,j}(t,T) = -\lambda_j(T)P_{i,j}(t,T) + \lambda_{j-1}(T)P_{i,j-1}(t,T) \ , \\ &\frac{d}{dT}P_{i,N}(t,T) = -\lambda_{N-1}(T)P_{i,N-1}(t,T) \end{aligned} \tag{26}$$

for all $i, j = 0, 1, \ldots, N$ and $0 \leq t \leq T$. The initial conditions are $P_{i,j}(t,t) = \mathbf{1}_{\{i=j\}}$. The solution of the Kolmogorov equations in equation (26) is as given below

$$P_{i,j}(t,T) = \begin{cases} 0 & \text{for} \quad i > j \\ \exp\{-\int_t^T \lambda_i(t,s)ds\} & \text{for} \quad i = j \\ \int_t^T P_{i,j-1}(t,s)\lambda_{j-1}e^{-\int_t^T \lambda_j(t,u)du}ds & \text{for} \quad i < j \end{cases} . \tag{27}$$

The representation of the implied loss distribution at time t is simply

$$p_i(t,T) = P[L(T) = i|\mathcal{F}_t] = P_{L(t),i}(t,T). \tag{28}$$

For example, if $L(t) = k$, then the implied loss distribution at time t is

$$p_i(t,T) = P_{k,i}(t,T). \tag{29}$$

6.3 Dynamic Loss Process

In the dynamics version of the loss process model, the loss process follows a Poisson process with time- and state-dependent inhomogeneous default intensities $\lambda_{L(t)}(t)$, $L(t) = 0, \ldots, N-1$, which are the transition rates in the generator matrix in equation (25). The aggregate default intensity $\lambda_{L(t)}(t)$ can be expressed in terms of the individual intensities $\lambda^k(t)$

$$\lambda_{L(t)}(t) = \sum_{k \in S(t)} \lambda^k(t), \tag{30}$$

where $S(t) := \{1 \leq k \leq N|\tau_k > t\}$ is the set of companies that have not defaulted by time t.

The loss process is assumed to follow a Poisson process with stochastic intensity, a process referred to as a *Cox process*.

$$d\lambda_i(t,T) = \mu_i(t,T)dT + \sigma_i(t,T)dB(t), \quad i = 0,\ldots,N-1, \tag{31}$$

where $B(t)$ is a *d-dimension Q-Brownian motion*, the $\mu_i(t,T)$'s are the drifts of the stochastic processes, and the $\sigma_i(t,T)$'s are the *d-dimension* volatilities of the stochastic processes. To keep the stochastic processes consistent with the loss process $L(t)$, the following conditions must be satisfied

$$P_{L(t),i}(t,T)\mu_i(t,T) = \sigma_i(t,T)\upsilon_{L(t),i}(t,T), \quad 0 \le i \le N-1, \quad t \le T, \tag{32}$$

where, $\upsilon_{n,m}(t,T)$'s are given by

$$\upsilon_{i,j} = \begin{cases} 0 & \text{for} \quad i > j \\ P_{nm}(t,T)\{-\int_t^T \sigma_i(t,s)ds\} & \text{for} \quad i = j \\ \int_t^T e^{-\int_s^T \lambda_j(t,u)du}[\sigma^{Pa}_{i,j-1}(t,s) - P_{ij}(t,s)\sigma_j(t,s)]ds & \text{for} \quad i < j \end{cases}, \tag{33}$$

with

$$\sigma^{Pa}_{i,j-1}(t,T) = P_{i,j-1}(t,T) + \lambda_{m-1}(t,T)\upsilon_{n,m-1}(t,T). \tag{34}$$

6.4 Default Correlation

In the loss process model, the default correlations between companies can arise from both the transition rates of the loss process and the volatilities of the stochastic processes. To understand the default dependence by the transition rates, recall the concept of default correlation. The default correlation is the phenomenon of joint defaults and a clustering of defaults. After one or more companies defaults, the individual default intensities of the surviving companies increase. The dependence of individual default intensities on the default number (loss process $L(t)$) can be reflected by a proper selection of the transition rates $\lambda_i(t), i = 0, 1, \ldots, N-1$. This is the way that the transition rates can cause the default dependence between companies.

The default dependence by the volatilities can be explained by considering the case of a one-dimension driving Brownian motion. For non-zero transition rate volatilities

$$\sigma_i(t, T) > 0 \quad \text{for all} \quad 0 \leq i \leq N-1, \tag{35}$$

Brownian motion works like an indicator of the common market condition. If its value is positive, the market condition is bad and all the transition rates are larger; if its value is negative, the market condition is good and all the transition rates are smaller.

6.5 Implementation of Dynamic Loss Process Model

The model can be implemented by a Monte Carlo method. For pricing a CDO with a maturity T, the procedure is as follows:

1. Initial condition: $t = 0$, $L(0) = 0$ ($p_0(0, 0) = 1$), and specify $\lambda_i(0, 0)$'s and $\sigma_i(0, .)$'s.

2. Simulate a Brownian motion trial.

3. $s \to s + \Delta s$: (until $s = T$)

    - Calculate $P_{0,m}(0, s)$ from equation (27), and $\upsilon_{0,j}(0, s)$ from equation (33), and use them and $\sigma_i(0, s)$ to calculate $\mu_i(0, s)$ from equation (32).
    - Calculate $\lambda_i(0, s + \Delta s)$ using the Euler scheme and μ_i(0,s)S and $\sigma_i(0, s)$.

- In a Euler scheme, calculate the loss distribution $p_i(0, s + \Delta s)$ from (27) and using the representation of the loss distribution in equation (29).
- The loss distribution $p_i(0, .)$ on the time period of $(0, T)$ is then calculated.

4. Repeat steps 2-4 until the average loss distributions $\overline{p}_i(0, .)$ of all the trials converge.

5. Using the average loss distributions $\overline{p}_i(0, .)$ to price a CDO.

The loss process model can also be used to price other portfolio credit derivatives such as basket default swaps, options on CDS indexes, and options on CDS indexes tranches.

7 Models for Pricing Correlation Risk

In this section, we give our suggestions for future research. It includes two parts. In the first part, we analyze the shortcoming of the one-factor double t copula model, and then propose four new heavy-tailed one-factor copula models. In the second part, we give our proposal for improving the structural model and the loss process model.

7.1 Heavy-Tailed Copula Models

Hull and White (2004) first use heavy-tailed distributions (Student's t distributions) in a one-factor copula model. In their so-called one-factor double t copula model, the degrees of freedom parameter of t distribution ν decreases, the tail-fatness of copula function increases, when the degrees of freedom parameter ν goes to infinity, the model becomes the one-factor Gaussian copula model.

As mentioned before, Hull and White find that the double t copula model fits market data well when the degrees of freedom parameter ν is equal to 4. But the simulation by Kalemanova et al. (2005) shows a different result. When Kalemanova et al. compare their model with the double t copula model, in addition to the simulation results by their own model, they also give the simulation results by the double t copula model for both the cases of the degrees of freedom parameter ν equal to 3 and 4. These simulation results show that the double t copula model fits market data better when $\nu = 3$ than $\nu = 4$. One difference in these two works is that different market data are used in the simulation. Hull and White use market data for the 5-year iTraxx Europe tranches on August 4, 2004, while Kalemanova et al. use market data on April 12, 2006. Therefore, the difference, related to how many degrees of freedom make the double t copula fit market data well, may suggest that for market data in different times, the double t copula model with different tail-fatnesses works well.

The drawbacks of the double t copula are that its tail fatness cannot be changed continuously and the maximum tail-fatness occurs when the degrees of freedom parameter ν equal to 3. In order to fit market data well over time, it is necessary that the tail-fatness of a one-factor copula model can be adjusted continuously and can be much larger than the maximum tail-fatness of the one-factor double t copula model.

In the following, we suggest four one-factor heavy-tailed copula models. Each model has (1) a tail-fatness parameter that can be changed continuously and (2) a maximum tail-fatness much larger than that of the one-factor double t copula model.

7.1.1 One-factor double mixture Gaussian copula model

The mixture Gaussian distribution is a mixture distribution of two or more Gaussian distributions. For simplicity, we consider the case of the mixture distribution of two Gaussian distributions which have a zero mean. If the random variable Y is such a mixture Gaussian distribution, then it can be expressed as

$$Y = \begin{cases} X_1 & \text{with probability} \quad p \\ X_2 & \text{with probability} \quad 1-p \end{cases}, \tag{36}$$

where $X1$ and $X2$ are independent normal Gaussian distributions with a zero mean

$$EX_1 = EX_2 = 0, \quad VarX_1 = \sigma_1^2 \quad \text{and} \quad VarX_2 = \sigma_2^2, \tag{37}$$

with $\sigma_1 > \sigma_2$. The mixture Gaussian distribution Y has a zero mean. Its variance is

$$VarY = p\sigma_1^2 + (1-p)\sigma_2^2. \tag{38}$$

The pdf of the distribution Y is

$$f_Y(y) = \frac{p}{\sqrt{2\pi}\sigma_1} \exp(-\frac{y^2}{2\sigma_1^2}) + \frac{1-p}{\sqrt{2\pi}\sigma_2} \exp(-\frac{y^2}{2\sigma_2^2}). \tag{39}$$

The mixture Gaussian distribution Y can be normalized by the following transition

$$\tilde{Y} = \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}} Y. \tag{40}$$

The pdf of $\tilde{Y}$ is

$$f_{\tilde{Y}}(y) = \begin{array}{l} \frac{p\sqrt{p\sigma_1^2+(1-p)\sigma_2^2}}{\sqrt{2\pi}\sigma_1} \exp(-\frac{y^2(p\sigma_1^2+(1-p)\sigma_2^2)}{2\sigma_1^2}) \\ +\frac{(1-p)\sqrt{p\sigma_1^2+(1-p)\sigma_2^2}}{\sqrt{2\pi}\sigma_2} \exp(-\frac{y^2(p\sigma_1^2+(1-p)\sigma_2^2)}{2\sigma_2^2}) \end{array}. \tag{41}$$

Using the standardized mixture Gaussian distribution in equation (41) as the distribution of the M and Z_i's in equation (3), we obtain our first extension to the one-factor Gaussian copula model which we refer to as a *double mixture Gaussian distribution copula model*. In this model, the tail-fatness of the M and Z's is determined by the parameters σ_1, σ_2, and p. In the implementation of the model, we can fix the parameters σ_1 and σ_2, and make the parameter p the only parameter to control the tail-fatness of the copula function.

7.1.2 One-factor double t distribution with fractional degrees of freedom copula model

The pdf of the gamma(α, β) distribution is

$$f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} exp(-x/\beta), \quad 0 < x < \infty, \quad \alpha > 0, \quad \beta > 0 \tag{42}$$

Setting $\alpha = \nu/2$ and $\beta = 2$, we obtain an important special case of the gamma distribution, the Chi-square distribution, which has the following pdf:

$$f(x|\nu) = \frac{1}{\Gamma(\nu/2)2^{\nu/2}} x^{\nu-1} exp(-x/2), \quad 0 < x < \infty, \quad \nu > 0. \tag{43}$$

If the degrees of freedom parameter ν is an integer, equation (43) is the Chi-square distribution with ν degrees of freedom. However, the degrees of freedom parameter ν need not be an integer. When ν is extended to a positive real number, we get the Chi-square distribution with ν fractional degrees of freedom.

If U is a standard normal distribution, V is a Chi-square distribution with ν fractional degrees of freedom, and U and V are independent, then $T = U/\sqrt{V/\nu}$

has the following pdf

$$f_T(t|\nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\nu\pi}}(1 + t^2/\nu)^{-(\nu+1)/2}, \quad 0 < x < \infty, \quad \nu > 0. \tag{44}$$

This is the Student's t distribution with ν fractional degrees of freedom (see Mardia and Zemroch (1978)). Its mean and variance are respectively

$$ET = 0, \qquad \nu > 1; \qquad VarT = \frac{\nu}{\nu - 2}, \qquad \nu > 2. \tag{45}$$

For $\nu > 2$, the Student's t distribution in equation (44) can be normalized by making the transition

$$X = \sqrt{(\nu - 2)/\nu}T, \quad \nu > 2. \tag{46}$$

The normalized Student's t distribution with $\nu(\nu > 2)$ factional degrees of freedom has the following pdf

$$f_X(x|\nu) = \sqrt{\frac{\nu}{\nu - 2}}\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\nu\pi}}(1 + \frac{x^2}{\nu - 2})^{-(\nu+1)/2}, \quad 0 < x < \infty, \quad \nu > 2. \tag{47}$$

Using the normalized Student's t distribution with factional degrees of freedom as the distribution of the M and Z_i's in equation (3), we get our second extension to the one-factor Gaussian copula model which we refer to as a *double t distribution with fractional degrees of freedom copula model.* In this model, the tail-fatness of the M and Z_i's can be changed continuously by adjusting the fractional degrees of freedom parameter ν.

7.1.3 One-factor double mixture distribution of t and Gaussian distribution copula model

In the previous model, the tail fatness of the M and Z_i's is controlled by the fractional degrees of freedom parameter of the Student's t distribution. Here, we introduce another distribution function for the M and Z_i's, the mixture distribution of the Student's t and the Gaussian distributions. Assume U is a normalized Student's t distribution with fractional degrees of freedom, and V is a standard normal distribution. We can express a mixture distribution X as

$$X = \begin{cases} U & \text{with probability} \quad 1-p \\ V & \text{with probability} \quad p \end{cases}, \qquad 0 \leq p \leq 1, \tag{48}$$

where p is the proportion of the Gaussian component in the mixture distribution X. The pdf of the X is

$$\begin{aligned} f(x) \quad = \quad & \tfrac{p}{\sqrt{2\pi}} exp(-x^2/2) + \\ & +(1-p)\sqrt{\tfrac{\nu-2}{\nu}} \tfrac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\nu/2)} (1 + \tfrac{x^2}{\nu-2})^{-(\nu+1)/2} \end{aligned}, \tag{49}$$

where ν is the fractional degrees of freedom of the Student's t distribution.

Using the mixture distribution of Student's t and Gaussian distributions in equation (3) as the distribution of the M and Z_i's, we get our third extension to the one-factor Gaussian copula model which we refer to as a *double mixture distribution of Student's t and Gaussian distribution copula model.* In this model, the tail-fatness of the M and Z's is controlled by the parameter p when the parameter ν is fixed.

7.1.4 One-factor double smoothly truncated stable copula model

In this part, we first introduce the stable distribution and the smoothly truncated stable distribution, and then provide our proposed model.

Stable distribution

A non-trivial distribution g is a stable distribution if and only if for a sequence of independent, identical random variables $X_{i,}, i = 1, 2, 3, \ldots, n$ with a distribution g, the constants $c_n > 0$ and d_n can always be found for any $n > 1$ such that

$$c_n(X_1 + X_2 + \cdots + X_n) + d_n \stackrel{d}{=} X_1.$$

In general, a stable distribution cannot be expressed in a closed form except for three special cases: Gaussian, Gauchy, and Lévy distributions. However, the characteristic function always exists and can be expressed in a closed form. For a random variable X with a stable distribution g, the characteristic function of the X can be expressed in the following form

$$\phi_X(t) = E\exp(itX) = \begin{cases} \exp(-\gamma^\alpha |t|^\alpha [1 - i\beta \text{sign}(t)\tan(\frac{\pi\alpha}{2})] + i\delta t), & \alpha \neq 1 \\ \exp(-\gamma|t|[1 + i\beta\frac{2}{\pi}\text{sign}(t)\ln(|t|)] + i\delta t), & \alpha = 1 \end{cases}, \tag{50}$$

where $0 < \alpha \leq 2$, $\gamma \geq 0$, $-1 \leq \beta \leq 1$, and $-\infty \leq \delta \leq \infty$, and the function of sign(t) is 1 when $t > 0$, 0 when $t = 0$, and -1 when $t < 0$.

There are four characteristic parameters to describe a stable distribution. They are: (1) the *index of stability* or the *shape* parameter α, (2) the *scale* parameter γ, (3) the *skewness* parameter β, and (4) the *location* parameter δ. A stable distribution g is called the α stable distribution and is denoted $S_\alpha(\delta, \beta, \sigma) = S(\alpha, \sigma, \beta, \delta)$.

The family of α stable distributions has three attractive properties:

- The sum of independent α stable distributions is still an α stable distribution, a property is referred to as *stability*.
- α stable distributions can be skewed.
- Compared with the normal distribution, α stable distributions can have a fatter tail and a high peak around its center, a property which is referred to as *leptokurtosis*.

Real world financial market data indicate that assets returns tend to be fat-tailed, skewed, and perked around center. For this reason α stable distributions have been a popular choice in modeling asset returns.[6]

Smoothly truncated α stable distribution

One inconvenience of a stable distribution is that it has an infinite variance except in the case of $\alpha = 2$. A new class of heavily-tailed functions is proposed by Menn and Rachev (2005): smoothly truncated α stable distribution.

A smoothly truncated α stable distribution is an α stable distribution with its two tails replaced by the tails of Gaussian distribution. The pdf can be expressed as

$$f(x) = \begin{cases} h_1(x) & for \quad x < a \\ g_\theta(x) & for \quad a \leq \delta \leq b \\ h_2(x) & for \quad x > b \end{cases} , \tag{51}$$

where $h_i(x), i = 1, 2$ are the pdf of two normal distributions with means μ_i and standard deviations σ_i, and $g_\theta(x)$ is the pdf of an α stable distribution with its parameter vector $\theta = (\alpha, \gamma, \beta, \delta)$. To secure a well-defined smooth probability distribution, the

[6] see Rachev, Menn, and Fabozzi (2005).

following regularities are imposed:

$$\begin{array}{l} h_1(a) = g_\theta(a), \qquad h_2(b) = g_\theta(b) \\ p_1 := \int_{-\infty}^{a} h_1(x)dx = \int_{-\infty}^{a} g_\theta(x)dx \\ p_2 := \int_{a}^{\infty} h_2(x)dx = \int_{b}^{\infty} g_\theta(x)dx \\ \sigma_1 = \frac{\psi(\varphi^{-1}(p_1))}{g_\theta(a)}, \qquad \mu_1 = a - \sigma_1\varphi^{-1}(p_1) \\ \sigma_2 = \frac{\psi(\varphi^{-1}(p_2))}{g_\theta(b)}, \qquad \mu_2 = b + \sigma_2\varphi^{-1}(p_2) \end{array}, \tag{52}$$

where ψ and φ denote the density and distribution functions of the standard normal distribution, respectively. A smoothly truncated α stable distribution is referred to as an *STS-distribution* and denoted by $S_\alpha^{[a,b]}(\gamma, \beta, \delta)$. The probabilities p_1 and p_2 are referred to as the *cut-off probabilities*. The real numbers a and b are referred to as the *cut-off points*.

The family of STS-distributions has two important properties. The first is that it is closed under the scale and location transitions. This means that if the distribution X is an STS-distribution, then for $c, d \in R$, the distribution $Y := cX + d$ is an STS-distribution. If X follows $S_\alpha^{[a,b]}(\gamma, \beta, \delta)$, then Y follows $S_{\tilde{\alpha}}^{[\tilde{a},\tilde{b}]}(\tilde{\gamma}, \tilde{\beta}, \tilde{\delta})$ with

$$\begin{array}{llll} \tilde{a} = ca + d, & \tilde{b} = cb + d, & \tilde{\alpha} = \alpha, & \\ \tilde{\gamma} = |c|\gamma, & \tilde{\beta} = \mathrm{sign}(c)\beta, & \tilde{\delta} = \begin{cases} c\delta + d & \alpha = 1 \\ c\delta - \frac{2}{\pi} c \log|c|\sigma\beta + d & \alpha \neq 1 \end{cases} \end{array}, \tag{53}$$

The other important property of the STS-distribution is that with respect to an α stable distribution $S_\alpha(\gamma, \beta, \delta)$, there is a unique normalized STS-distribution $\tilde{S}_\alpha^{[a,b]}(\gamma, \beta, \delta)$ whose cut-off points a and b are uniquely determined by the four parameters α, γ, β, and δ. Because of the uniqueness of cut-off points, the normalized STS-distribution can be denoted by the *NSTS-distribution* $\tilde{S}_\alpha(\gamma, \beta, \delta)$.

One-factor double smoothly truncated stable copula model

In the one-factor copula model given in equation (3), using the NSTS-distribution $\tilde{S}_\alpha(\gamma, \beta, \delta)$ for the distribution of the market component M and the individual components Z_i's, we obtain the fourth extension to the one-factor Gaussian copula model. We refer to the model as a *one-factor double smoothly truncated α stable copula model*. In the model, we can fix the parameters γ, β, and δ, and make the parameter α the only parameter to control the tail fatness of the copula function. When the parameter $\alpha = 2$, the model becomes the one-factor Gaussian copula model. When α decreases, the tail-fatness increases.

7.2 Suggestions for Structural Model and Loss Process Model

The base-case structural model suggested by Hull et. al (2005) can be an alternative method to the one-factor Gaussian copula model. The results of the two models are close. Consider the fact that the one-factor double t copula model fits market data much better than the one-factor Gaussian copula model according to Hull and White (2004). A natural way to enhance the structural model is by applying heavy-tailed distributions.

Unlike the one-factor copula model, where any continuous distribution with a zero mean and a unit variance can be used, in the structural model there is a strong constraint imposed on the distribution of the underlying stochastic processes. The distribution for the common driving process $M(t)$ and the individual driving process Z_i's in equation (22) must satisfy a property of closure under summation. This means that if two independent random variables follow a given distribution, then the sum of these two variables still follow the same distribution. As explained earlier, the α stable distribution has this property and has been used in financial modeling.[7]

[7] see Rachev and Mitnik (2000).

We suggest using the α stable distribution in the structural model.

The non-Gaussian α stable distribution has a drawback. Its variance does not exist. The STS distribution is a good candidate to overcome this problem. For a STS distribution, if the two cut-off points a and b are far away from the peak, the STS distribution is approximately closed under summation. Based on this, employing the STS distribution in the structural model should be the subject of future research.

In the dynamic loss process model, the default intensities λ_i's follow stochastic processes as shown in equation (31). It also a possible research direction to use the α stable distribution and the STS distribution for the driving processes.

8 Summary

In this paper, we review three models for pricing portfolio risk: the one-factor copula model, the structural model, and the loss process model. We then propose how to improve these models by using heavy-tailed functions. For the one-factor copula model, we suggest using (1) a double mixture Gaussian copula, (2) a double t distribution with fractional copula, (3) a double mixture distribution of t and Gaussian distributions copula, and (4) a double smoothly truncated α stable copula. In each of these four new extensions to the one-factor Gaussian copula model, one parameter is introduced to control the tail-fatness of the copula function. To improve the structural and loss process models, we suggest using the stable distribution and the smoothly truncated stable distribution for the underlying stochastic driving processes.

References

[1] Amato J, Gyntelberg J (2005) CDS index tranches and the pricing of credit risk correlations. BIS Quarterly Review, March 2005, pp 73-87

[2] Bennani N (2005) The forward loss model: a dynamic term struture approach for the pricing of porftolio credit derivatives. Working paper, available at *http://www.defaultrisk.com/pp_crdrv_95.htm*

[3] Black F, Cox J (1976) Valuing corporate securities: some effects of bond indenture provision. The Journal of Finance, vol 31, pp 351-367

[4] Di Graziano G, Rogers C (2005) A new approach to the modeling and pricing of correlation credit derivatives. Working paper, available at *www.defaultrisk.com/pp_crdrv_88.htm*

[5] Duffie D (2004) Comments: irresistible reasons for better models of credit risk. Financial Times, April 16, 2004

[6] Hull J, White A (2004) Valuation of a CDO and nth to default CDS without monte carlo simulation. The Journal of Derivatives, vol 2, pp 8-23

[7] Hull J, Predescu M, White A (2005) The valuation of correlation-dependent credit derivatives using a structural model. Working Paper, Joseph L. Rotman School of Management, University of Toronto, available at *http://www.defaultrisk.com/pp_crdrv_68.htm*

[8] Kalemanova A, Schmid B, Werner R (2005) The normal inverse gaussian distribution for synthetic CDO pricing. Working paper, available at *http://www.defaultrisk.com/pp_crdrv_91.htm*

[9] Laurent JP, Gregory J (2003) Basket default swaps, CDOs and factor copulas. Working paper, ISFA Actuarial School, University of Lyon, available at *http://www.defaultrisk.com/pp_crdrv_26.htm*

[10] Li DX (1999) The valuation of basket credit derivatives. CreditMetrics Monitor, April 1999, pp 34-50

[11] Li DX (2000) On default correlation: a copula function approach. The Journal of Fixed Income, vol 9, pp 43-54

[12] Lucas DJ, Goodman LS, Fabozzi FJ (2006) Collateralized debt obligations: structures and analysis, 2nd edn. Wiley Finance, Hoboken, New Jersey

[13] Mardia K, Zemroch P (1978) Tables of the F- and related distributions with algorithms. Academic Press, New York

[14] McGinty L, Beinstein E, Ahluwalia R, Watts M (2004) Credit correlation: a guide. Credit Derivatives Strategy, JP Morgan, London, March 12, 2004

[15] Menn C, Rachev S (2005) A GARCH option pricing model with alpha-stable innovations. European Journal of Operational Research, vol 163, pp 201-209

[16] Menn C, Rachev S (2005) Smoothly truncated stable distributions, Garch-models, and option pricing. Working paper, University of Karlsruhe and UCSB, available at *http://www.statistik.uni-karlsruhe.de/download/tr_smoothly_truncated.pdf*

[17] Merton R (1974) On the pricing of corporate debt: the risk structure of interest rates. The Journal of Finance, vol 29, pp 449-470

[18] Rachev S, Mitnik S (2000) Stable paretian models in finance. John Wiley, Series in Financial Ecomomics and Quatitative Analysis, Chichester.

[19] Rachev S, Menn C, Fabozzi FJ (2005) Fat-tailed and skewed asset return distributions: implications for risk management, Portfolio selection, and Option Pricing. Wiley Finance, Hoboken, New Jersey.

[20] Schönbucher P (2005) Portfolio losses and the term structure of loss transition rates: a new methodology for the pricing of portfolio credit derivatives. Working paper, available at *http://www.defaultrisk.com/pp_model_74.htm*

[21] Sidenius J, Piterbarg V, Andersen L (2005) A new framework for dynamic credit portfolio loss modeling. Working paper, available at *http://www.defaultrisk.com/pp_model_83.htm*