

# The Proper Use of Risk Measures in Portfolio Theory

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## Abstract

This paper discusses and analyzes risk measure properties in order to understand how a risk measure has to be used to optimize the investor's portfolio choices. In particular, we distinguish between two admissible classes of risk measures proposed in the portfolio literature: safety risk measures and dispersion measures. We study and describe how the risk could depend on other distributional parameters. Then, we examine and discuss the differences between statistical parametric models and linear fund separation ones. Finally, we propose an empirical comparison among three different portfolio choice models which depend on the mean, on a risk measure, and on a skewness parameter. Thus, we assess and value the impact on the investor's preferences of three different risk measures even considering some derivative assets among the possible choices.

*Key words:* skewness, safety risk measures, risk aversion, dispersion measures, portfolio selection, investors' preference, fund separation.

*JEL Classification:* G14, G15

## 1. INTRODUCTION

Many possible definitions of risk have been proposed in the literature because different investors adopt different investment strategies in seeking to realize their investment objectives. In some sense risk itself is a subjective concept and this is probably the main characteristic of risk. Thus, even if we can identify some desirable features of an investment risk measure, probably no unique risk measure exists that can be used to solve every investor's problem. Loosely speaking, one could say that before the publication of the paper by Artzner, Delbaen, Eber, and Heath (2000) on coherent risk measures, it was hard to discriminate between "good" and "bad" risk measures. However, the analysis proposed by Artzner, *et al.*(2000) was addressed to point out the value of the risk of future wealth, while most of portfolio theory has based the concept of risk in strong connection with the investor's preferences and their "utility function".

From an historical point of view, the optimal investment decision always corresponds to the solution of an "expected utility maximization problem". Therefore, although risk is a subjective and relative concept (see Balzer (2001), Rachev *et al* (2005)) we can always state some common risk characteristics in order to identify the optimal choices of some classes of investors, such as non-satiable and/or risk-averse investors. In particular, the link between expected utility theory and the risk of some admissible investments is generally represented by the consistency of the risk measure with a stochastic order.<sup>1</sup>

Thus, this property is fundamental in portfolio theory to classify the set of admissible optimal choices. On the other hand, there exist many other risk properties that could be used to characterize investor's choices. For this reason, in this paper, we classify several risk measure properties for their financial insight and then discuss how these properties characterize the different use of a risk measure.

In particular, we describe three risk measures (MiniMax, mean-absolute deviation, and standard deviation) and we show that these risk measures (as many others) can be

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<sup>1</sup> Recall that the wealth  $X$  first order stochastically dominates the risky wealth  $Y$  ( $X$  FSD  $Y$ ) if and only if for every increasing utility function  $u$ ,  $E(u(X)) \geq E(u(Y))$  and the inequality is strict for some  $u$ . Analogously, we say that  $X$  second order stochastically dominates  $Y$  ( $X$  SSD  $Y$ ), if and only if for every increasing, concave utility functions  $u$ ,  $E(u(X)) \geq E(u(Y))$  and the inequality is strict for some  $u$ . We also say that  $X$  Rothschild Stiglitz stochastically dominates  $Y$

considered equivalent by risk-averse investors, although they are formally different. Then we discuss the multi-parameter dependence of risk and show how we could determine the optimal choices of non-satiable and/or risk-averse investors. In particular, we observe that when asset returns present heavy tails and asymmetries, fund separation does not hold. However, if we consider the presence of the riskless asset, then two fund separation holds among portfolios with the same skewness and kurtosis parameters. Finally, we propose an empirical comparison among different portfolio allocation problems in a three parameter context in order to understand the impact that MiniMax, mean-absolute deviation, and standard deviation could have for some non-satiable and risk-averse investors. In this framework we also consider the presence of some contingent claims and compare the optimal choices of several investors in a mean-risk-skewness space.

## 2. RISK MEASURES AND THEIR PROPERTIES

Let us consider the problem of optimal allocation among  $n$  assets with vector of returns

$r=[r_1, \dots, r_n]'$  where  $r_i = \frac{P_{i,t+1} - P_{i,t}}{P_{i,t}}$  while  $P_{i,t}$  is the price of  $i$ -th asset at time  $t$ . No short

selling is allowed, i.e., the wealth  $y_i$  invested in the  $i$ -th asset is non negative for every  $i=1, \dots, n$ . Thus considering an initial wealth  $W_0$ , imagine that the following optimization problem:

$$\begin{aligned} \min_y & p(W_0 + y'r) \\ \sum_{i=1}^n y_i &= W_0 \quad y_i \geq 0 \quad i = 1, \dots, n \\ E(W_0 + y'r) &\geq \mu_y \end{aligned} \tag{1}$$

is equivalent to maximizing the expected utility  $E(U(W_y))$  of the future wealth

$W_y := W_0 + y'r$  invested in the portfolio of assets. Then, we implicitly assume that the

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(X R-S Y) if and only if for every concave utility functions  $u$ ,  $E(u(X)) \geq E(u(Y))$  and the inequality is strict for some  $u$ . (See, among others, Levy (1992) and the references therein).

expected utility of the future wealth  $W_y$ , has a mean greater than  $\mu_y$  and the expected utility depends only on the mean and the risk measure  $p$ . In this case, we say that the risk measure  $p$  is *consistent* with the order relation induced by the utility function  $U$ .

More generally, a risk measure is *consistent* with an order relation (Rothschild-Stiglitz stochastic order, first-order stochastic dominance, second-order stochastic dominance) if  $E(U(W_x)) \geq E(U(W_y))$  (for all utility functions  $U$  belonging to a given category of functions: increasing; concave; increasing and concave) implies that  $p(W_x) \leq p(W_y)$  for all admissible future wealths  $W_x, W_y$ . Consistency is absolutely necessary for a risk measure to make sense. It ensures us that we can characterize the set of all the optimal choices when either wealth distributions or expected utility depend on a finite number of parameters.<sup>2</sup> Although, when we assume that either wealth distributions or expected utility depend on more than two parameters (the mean, the risk, and other skewness and/or kurtosis parameters – see Section 4), the complexity of the optimization problem could increase dramatically. As a consequence of consistency, all the best investments of a given category of investors (non-satiable, risk-averse, non-satiable and risk-averse) are among the less risky ones. But the converse is not generally true; that is, we cannot guarantee that all the less risky choices are the best ones even if the risk measure is consistent with some stochastic orders. In fact, any risk measure associates only a real number to a random wealth, while the stochastic orders compare all cumulative distribution functions. Then, intuitively, a unique number cannot summarize the information derived from the whole wealth distribution function.

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<sup>2</sup> See Ortobelli (2001).

This is the main reason why every risk measure is incomplete and other parameters have to be considered. The standard deviation

$$STD(W_y) = E\left(\left(W_y - E(W_y)\right)^2\right)^{1/2} \quad (2)$$

is the typical example of a risk measure consistent with Rothschild-Stiglitz (R-S) stochastic order (concave utility functions). It was also the first measure of uncertainty proposed in portfolio theory for controlling portfolio risk (see Markowitz (1952-1959) and Tobin (1958)). Another example of a risk measure consistent with Rothschild-Stiglitz stochastic order is the *mean-absolute deviation* (MAD)

$$MAD(W_y) = E\left(|W_y - E(W_y)|\right), \quad (3)$$

where the risk is based on the absolute deviations from the mean rather than the squared deviations as in the case of the standard deviation. The MAD is more robust with respect to outliers and proposed as a measure to order the investor's choices (see Konno and Jamazaki (1991), Speranza (1993), and Ogryczak and Ruszczyński (1999)).

Artzner *et al* (2000) have defined another type of consistency, called *monotony*, that is  $p(W_y) \leq p(W_x)$  for the risky wealths  $W_y$  and  $W_x$  that satisfy  $W_y \geq W_x$ .

Sometimes there is only a partial consistency between a risk measure and a stochastic order. For example, we say that a risk measure is *consistent with first-order stochastic dominance with respect to additive shifts* if  $p(W_x) \leq p(W_y)$  when  $W_x = W_y + t$ , for some constant  $t \geq 0$ . In this case, the wealth  $W_x$  is considered less risky than  $W_y$  by any investor that prefers more than less. An example of a monotone risk measure proposed by Young (1998) for portfolio theory that is consistent with first and second order stochastic dominance is the *MiniMax* (MM) risk measure,

$$MM(W_y) = -\sup\left\{c \in R \mid P(W_y \leq c) = 0\right\}. \quad (4)$$

Considering and realizing that the utility maximization problem can be difficult to solve, many researchers have sought and proposed equivalent formulations with nicer

numerical properties. This leads to the definition of the following properties. The first property which should be fulfilled by a risk measure is *positivity*. Either there is risk, this means  $p(W_y) > 0$  or there is no risk ( $p(W_y) = 0$ ). Negative values (less risk than no risk) does not make sense. Particularly, we impose the condition that  $p(W_y) = 0$  holds if and only if investment  $W_y$  is non-stochastic. This property is called *positivity*.

Clearly, different risk measures could have a different impact on the complexity of the problem given by (1). In particular, we must take into account the computational complexity when solving large-scale portfolio selection problems. Under some circumstances, it might happen that the resulting minimization problem might be linearizable, which implies easy solution algorithms; in this case, we call the risk measure *linearizable*. Hence, the success of some risk measures is due to the computational practicability of the relative linearizable optimization problems.

Another important property which should be accounted for by the risk measure is the effect of diversification: if the wealth  $W_y$  bears risk  $p(W_y)$  and investment  $W_x$  bears risk  $p(W_x)$ , then the risk of investing half of the money in the first portfolio and half of the money in the second one should be not be greater than the corresponding weighted sum of the risks. Formally, we have:  $p(\lambda W_x + (1 - \lambda)W_y) \leq \lambda p(W_x) + (1 - \lambda)p(W_y)$  for all  $\lambda \in [0, 1]$ . A risk measure  $p$  fulfilling this equation is called *convex*. The property of convexity can also be deduced if the risk measure fulfills two other properties which are called *subadditivity* and *positive homogeneity*:

(1)  $p$  is subadditive if  $p(W_x + W_y) \leq p(W_x) + p(W_y)$  and

(2) it is called *positive homogeneous* if  $p(\alpha W_x) = \alpha p(W_x)$  for all random wealth  $W_x$  and real  $\alpha > 0$ .

The last property of risk measures is called *translation invariance*. There are different definitions of translation invariance. We obtain the so-called *Gaivoronsky-Pflug* (G-P) *translation invariance* (see Gaivoronsky and Pflug (2001)) if for all real  $t$ :

$p(W_x+t)=p(W_x)$ . This property can be interpreted as follows: the risk of a portfolio cannot be reduced or increased by simply adding a certain amount of riskless money. This property is, for example, fulfilled by the standard deviation but not fulfilled by the MiniMax measure or the Conditional Value at Risk (CVaR) measure that has recently been suggested for risk management. Alternatively, translation invariance holds if  $p(W_x+t) = p(W_x)-t$  for all real  $t$ . Furthermore, we can generalize all the previous definitions of translation invariance considering the so-called *functional translation invariance*, if for all real  $t$  and any risky wealth  $W_x$ , the function  $f(t)=p(W_x+t)$  is a continuous and non-increasing function. This property summarizes not only the different definitions of translation invariance, but it considers also the consistency with first-order stochastic dominance with respect to additive shifts.

In order to take into account the temporal dependence of risk, the above static properties can be generalized to an intertemporal framework assuming the same definitions at each moment of time (see, among others, Artzner *et al* (2003)). Artzner, *et al* (2000) have called a *coherent* risk measure any translation invariant, monotonous, subadditive, and positively homogeneous risk measure. In particular the MiniMax measure can be seen as an extreme case of conditional value at risk (CVaR), that is a coherent risk measure. Other risk measure classifications have been proposed recently. In particular, Rockafeller *et al* (2003) (see also Ogryczak and Ruszczyński (1999)) define *deviation measure* as a positive, subadditive, positively homogeneous, G-P translation invariant risk measure and *expectation-bounded risk measure* as any translation invariant, subadditive, positively homogeneous risk measure  $p$  that associates the value  $p(W_x) > -E(W_x)$  with a non-constant wealth  $W_x$ . Typical examples of deviation measures are the standard deviation given by (2) and the MAD given by (3), while the MiniMax measure given by (4) is a coherent expectation-bounded risk measure. The most important feature of these new classifications is that there exists a corresponding one-to-one relationship between deviation measures and expectation-bounded risk measures.

As a matter of fact, given a deviation measure  $p$ , then the measure defined  $q(W_x) =$



$p(W_x) - E(W_x)$  for any risky wealth  $W_x$  is an expectation-bounded risk measure. Conversely, given an expectation-bounded risk measure  $q$ , then the measure defined  $p(W_x) = q(W_x - E(W_x))$  is a deviation measure. Thus, the deviation measure associated with MiniMax is given by  $MM(W_x - E(W_x))$ .

### 3. MEASURES OF UNCERTAINTY AND PROPER RISK MEASURES

From the discussion above, some properties are substantially in contrast with others. For example, it is clear that a G-P translation invariant measure cannot be translation invariant and/or consistent with first-order stochastic dominance (FSD) due to additive shifts. As a matter of fact, G-P translation invariance implies that the addition of certain wealth does not increase the uncertainty. Thus, this concept is linked to uncertainty. Conversely the translation invariance and consistency with FSD due to additive shifts imply that the addition of certain wealth decreases the wealth under risk even if it does not increase uncertainty.

Artzner *et al* (2000) have identified in the coherent property “the right price” of risk. However, in the previous analysis, we have identified some properties which are important to measure the *uncertainty* and other properties which are typical of the *proper risk measures* because they are useful to value wealth under risk. Clearly, coherency is typical of proper risk measures. Instead, a positive risk measure  $p$  does not distinguish between two certain wealths  $W_1$  and  $W_2$  because  $p(W_1) = p(W_2) = 0$  even if  $W_1 < W_2$  and the second wealth is preferred to the first one. That is, if wealth  $W_x$  presents uncertainty, then  $p(W_x) > 0$ , otherwise no uncertainty is allowed and  $p(W_x) = 0$ .

We meet an analogous difference between the two categories of risk measures if we consider the risk perception of different investors. So, risk aversion characterizes investors who want to limit the uncertainty of their wealth. Instead, non-satiable investors want to increase wealth, thus they implicitly reduce the wealth under risk. Therefore, the consistency with Rothschild-Stiglitz stochastic order is typical of uncertainty measures and the consistency with FSD order or the monotony characterizes the proper risk measures.

In contrast, there are some properties that are useful in order to measure uncertainty and wealth under risk. For example, convexity is a property that identifies the importance of diversification. Undiversified portfolios present a greater grade of uncertainty and a larger wealth under risk. Similarly, positive homogeneity implies that when wealth under risk is multiplied by a positive factor, then risk and uncertainty must also grow with the same proportionality. In addition, it is possible to show that positivity, functional translation invariance, and positive homogeneity are sufficient to characterize the uncertainty of any reasonable family of portfolio distributions.<sup>3</sup> Thus, we will generally require that at least these properties are satisfied by any uncertainty measure. Moreover, considering that consistency is the most important property in portfolio theory, we require that any measure of wealth under risk is at least consistent with FSD.

Table 1 summarizes the properties of uncertainty measures and proper risk measures of wealth under risk. However, this classification is substantially known in the literature despite the fact that researchers have labeled the two categories of risk measures differently and have not identified all their properties and characteristics. As a matter of fact, according to the portfolio theory literature, we can define these two disjoint categories of risk measures as dispersion measures and safety-risk measures. Typically, a dispersion measure values the grade of uncertainty, and a safety-first measure values wealth under risk. In very general terms, we say that a dispersion measure is a strictly increasing function of a functional translation invariant, positive and positively homogeneous risk measure, while a safety-risk measure is consistent with FSD. The two categories are disjointed since a dispersion measure is never consistent with FSD. More precisely, given a positive risky wealth  $W_x$  and a positive  $\alpha < 1$ , then it is not difficult to verify that  $W_x \text{ FSD } \alpha W_x$ . Thus, any safety-first risk measure  $q$  presents less risk for the dominant random variable, that is  $q(W_x) \leq q(\alpha W_x)$ . In contrast, a dispersion measure  $p$  is a strictly increasing function of a positive and positively homogeneous risk measure  $p_1$ , that is  $p = f(p_1)$  with  $f$  strictly increasing function. Therefore,  $p$  satisfies the relation

$$p(W_x) = f(p_1(W_x)) > f(\alpha p_1(W_x)) = p(\alpha W_x).$$

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<sup>3</sup> See Ortobelli (2001).

In particular, Tables 2 and 3 recall the definitions and the properties of some of the dispersion measures and safety-first risk measures proposed in the portfolio literature (for a review, see also Giacometti and Ortobelli (2004)).

Observe that  $X$  dominates  $Y$  for a given stochastic order (Rothschild-Stiglitz stochastic order, first-order stochastic dominance, and second-order stochastic dominance), if and only if  $\alpha X + b$  dominates  $\alpha Y + b$  for the same stochastic order, for any positive  $\alpha$  and real  $b$ . This is the main reason why we can interchange wealth and return in problems of type (1) with consistent risk measures. Let us refer to  $A$  as the class of optimal choices that we obtain solving the optimization problem (1) and varying  $\mu_y$  for a given consistent risk measure  $p$ . Then, the class  $A$  is practically the same (up to an affine transformation) to the one that we obtain by solving the same problem but considering either  $W_y/W_0$  or  $W_y/W_0 - 1$  instead of the final wealth  $W_y := W_0 + y'r$ . In

this case, the variables are the portfolio weights  $x_i = \frac{y_i}{W_0}$  ( $i=1, \dots, n$ ) that represent the percentage of wealth invested in the  $i$ -th asset. Besides, the future wealth of one unit invested today is given by  $1+x'r$ . Thus, the optimization problem (1) can be rewritten as:

$$\begin{aligned} \min_x & p(1+x'r) \\ \text{s.t.} & \sum_i x_i = 1, x_i \geq 0 \\ & E(1+x'r) \geq m_x \end{aligned} \tag{1'}$$

for an opportune level  $m_x$ . For this reason, in the following we deal and study simplified selection problems with the gross returns  $1+x'r$ , instead of the final wealth  $W_y := W_0 + y'r$ .

## 4. LIMITS AND ADVANTAGES OF RISK MEASURES IN PORTFOLIO OPTIMIZATION

### 4.1 How to Use Uncertainty Measures

In the previous analysis, we explained that the most widely used risk measure, the variance, is in reality a measure of uncertainty. Thus, the question is: When and how can we use an uncertainty measure to minimize risk?

When we minimize the risk measure at a fixed mean level, we are not trying to increase our future wealth (because the mean is fixed), but we are only limiting the uncertainty of future wealth. Thus we can obtain a portfolio that could be optimal for a risk-averse investor, but not necessarily for a non-satiable one. However, we do not have to minimize uncertainty in order to minimize risk. For example, suppose that future wealth is uniquely determined by the mean and a dispersion measure  $p$ . Assuming that no short sales are allowed, every non-satiable investor will choose a portfolio among the solutions of the following problem.<sup>4</sup>

$$\begin{aligned}
 & \max_x p(1 + x'r) \\
 & s.t. \sum_i x_i = 1, x_i \geq 0 \\
 & \frac{1 + \sum_i x_i E(r_i)}{p(1 + x'r)} = h
 \end{aligned} \tag{5}$$

where the ratio between the mean and the uncertainty measure must be greater than an opportune level  $h$ . That is, we maximize the uncertainty for an opportune level of wealth under risk. The level of wealth under risk is measured assuming that the expected future wealth is proportional to its uncertainty, i.e.  $1 + \sum_i x_i E(r_i) = hp(1 + x'r)$ . Therefore, even if returns are uniquely determined from the mean and the variance, there are some optimal portfolios from the Markowitz' point of view which cannot be considered optimal for a non-satiable investors. In fact, Markowitz' analysis is theoretically justified only if distributions are unbounded elliptical (normal, for example) or investors have quadratic utility functions.

Figure 1 shows the optimal choices in a mean-dispersion plane. All the admissible choices have mean and dispersion in the closed area. In Figure 1, we implicitly assume

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<sup>4</sup> See Ortobelli (2001).

that future wealth is positive because wealth is not unbounded from below (in the worst case it is equal to zero when we lose everything). Thus, it is generally unrealistic to assume return distributions that are unbounded from below such as the normal one. Portfolios on the arc EA (in a neighborhood of the global minimum dispersion portfolio) are not optimal because there are other ones with greater uncertainty that are preferred by every non-satiable investor. Observe that the quadratic utility is not always increasing and it displays the undesirable satiation property. Thus, an increase in wealth beyond the satiation point decreases utility. Then, there could exist some quadratic utility functions whose maximum expected utility is attained at portfolios in the arc EA, but for any increasing utility function, the expected utility of portfolios on the arc EA is lower than the expected utility of some portfolios on the arc AB. From this example we see that although dispersion measures are uncertainty measures, we can opportunely use them in order to find optimal choices for a given class of investors. Moreover, minimum dispersion portfolios are not always optimal for non-satiable investors.

#### **4.2 Two Fund Separation and Equivalence Between Risk Measures**

Generally, we say that two risk measures are considered *equivalent* by a given category of investors if the corresponding mean-risk optimization problems generate one and the same solution. From the analysis of risk measure properties, we cannot deduce if there exists “the best” risk measure. In fact, under some distributional assumptions, it has been proven that all dispersion measures are equivalent. In particular, when we assume that choices depend on the mean and a G-P translation invariant, positive and positively homogeneous risk measure, then any other G-P translation invariant positive and positively homogeneous risk measure differs from the first one by a multiplicative positive factor.<sup>5</sup> This result implies that it in theory one is indifferent when deciding to employ one or any other existing G-P translation invariant positive and positively homogeneous risk measure (in a mean-risk framework). Furthermore, considering the equivalence between expectation-bounded risk measures and deviation measures, we have to expect the same results minimizing either a G-P translation invariant dispersion

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<sup>5</sup> See Ortobelli (2001).

measure, or an expectation-bounded safety risk measure for any fixed mean level.<sup>6</sup> Thus expectation-bounded safety risk measures are equivalent to the G-P translation invariant dispersion measures from the perspective of risk-averse investors. However, a comparison among several allocation problems, which assume various equivalent-risk measures, has shown that there exist significant differences in the portfolio choices.<sup>7</sup> There are two logical consequences of these results.

First, practically, the portfolio distributions depend on more than two parameters and optimal choices cannot be determined only by the mean and a single risk measure. This is also confirmed by empirical evidence. Return series often show “distributional anomalies” such as heavy tails and asymmetries. Then, it could be that different risk measures penalize/favor the same anomalies in a different way. For this reason, it makes sense to identify those risk measures that improve the performance of investors’ strategies.

Second, most of the mean-variance theory can be extended to other mean G-P translation invariant dispersion models and/or mean-expectation-bounded risk models. On the other hand, assume that the portfolio returns are uniquely determined by the mean and a G-P translation invariant positive and positively homogeneous risk measure  $\sigma_{x'r}$ . Thus, we obtain an analogous capital asset pricing model (CAPM) for any opportune mean-risk parameterization of the portfolio family. In particular, we can use the *extended Sharpe measure*  $\frac{E(x'r - r_0)}{\sigma_{x'r}}$  to identify superior, ordinary, and inferior performance of portfolio excess return  $x'r - r_0$  where  $r_0$  is the riskless return. If  $\bar{x}'r$  is the risky portfolio which maximizes the extended Sharpe measure, then, for any  $\lambda \in (0,1)$ , an optimal portfolio with the same mean and lower risk than  $z = \lambda r_0 + (1 - \lambda)\bar{x}'r$  cannot exist because  $\frac{E(z - r_0)}{\sigma_z} = \frac{E(\bar{x}'r - r_0)}{\sigma_{\bar{x}'r}}$ . Therefore, the portfolios  $r_0$  and  $\bar{x}'r$  span the efficient frontier and two fund separation holds. However, as it follows from the next

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<sup>6</sup> See Rockafeller et al. (2003), Ogryczak and Ruszczyński (1999), and Tokat et al (2003).

<sup>7</sup> See Giacometti and Ortobelli (2004).

discussion, we cannot generally guarantee that  $k$  fund separation holds when the portfolio of returns depend on  $k$  statistical parameters.

### 4.3 Multi-parameter Efficient Frontiers and Non-linearity

To take into account the distributional anomalies of asset returns, we need to measure the skewness and kurtosis of portfolio returns. In order to do this, statisticians typically use the so called Pearson-Fisher skewness and kurtosis indexes which provide a measure of the departure of the empirical series from the normal distribution. A positive (negative) index of asymmetry denotes that the right (left) tail of the distribution is more elongated than that implied by the normal distribution. The Pearson-Fisher coefficient of skewness is given by

$$\gamma_1(x'r) = \frac{E\left((x'r - E(x'r))^3\right)}{\left(E\left((x'r - E(x'r))^2\right)\right)^{3/2}}$$

The Pearson-Fisher kurtosis coefficient for a Gaussian distribution is equal to 3. Distributions whose kurtosis is greater (smaller) than 3 are defined as leptokurtic (platikurtic) and are characterized by fat tails (thin tails). The Pearson-Fisher kurtosis coefficient is given by

$$\gamma_2(x'r) = \frac{E\left((x'r - E(x'r))^4\right)}{\left(E\left((x'r - E(x'r))^2\right)\right)^2}.$$

According to the analysis proposed by Ortobelli (2001), it is possible to determine the optimal choice for an investor under very weak distributional assumptions. For example, when all admissible portfolios of gross returns are uniquely determined by the first  $k$  moments, under institutional restrictions of the market (such as no short sales and limited liability), all risk-averse investors optimize their portfolio choosing among the solutions of the following constrained optimization problem:

$$\begin{aligned}
& \min_x p(1+x'r) \quad \text{subject to} \\
& x'E(r) = m; \sum_i x_i = 1; x_i \geq 0 \quad i = 1, \dots, n \\
& \frac{E((x'r - E(x'r))^i)}{(x'Qx)^{i/2}} = q_i; \quad i = 3, \dots, k
\end{aligned} \tag{6}$$

for some mean  $m$  and  $q_i \quad i=3, \dots, k$ , where  $p(1+x'r)$  is a given dispersion measure of the future portfolio wealth  $1+x'r$  and  $Q$  is the variance–covariance matrix of the return vector  $r = [r_1, \dots, r_n]'$ . Moreover, all non-satiable investors will choose portfolio weights, solutions of the following optimization problem

$$\begin{aligned}
& \max_x p(1+x'r) \quad \text{subject to} \\
& \frac{1+x'E(r)}{p(1+x'r)} \geq h; \sum_i x_i = 1; x_i \geq 0 \quad i = 1, \dots, n; \\
& \frac{E((x'r - E(x'r))^i)}{(x'Qx)^{i/2}} = q_i \quad i = 3, \dots, k
\end{aligned} \tag{7}$$

for some  $q_i \quad i=3, \dots, k$ , and an opportune  $h$ . Similarly, all non-satiable risk-averse investors will choose portfolio weights that are solutions to the following optimization problem

$$\begin{aligned}
& \max_x E(x'r) \quad \text{subject to} \\
& \frac{1+x'E(r)}{p(1+x'r)} \geq h; \sum_i x_i = 1; x_i \geq 0 \quad i = 1, \dots, n; \\
& \frac{E((x'r - E(x'r))^i)}{(x'Qx)^{i/2}} = q_i \quad i = 3, \dots, k
\end{aligned} \tag{8}$$

for some  $q_i \quad i=3, \dots, k$ , and an opportune  $h$ . Moreover, in solving the above constrained problems, we can identify the optimal choices respect to other investor's attitude. As a matter of fact, it has been argued in the literature that decision makers have ambiguous skewness attitudes, while others say that investors are skewness-prone or prudent.<sup>8</sup>

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<sup>8</sup> See, among others, Horvarth and Scott (1980), Gamba and Rossi (1998), and Pressacco and Stucchi (2000).



For example, according to the definition given in Kimball (1990), we can recognize the nonsatiable, risk-averse investors who display *prudence*, i.e. the agents that display a skewness preference for fixed mean and dispersion. In any case, if we assume the standard deviation as a risk measure, we find that the Markowitz mean-standard deviation frontier is contained in the set of the solutions to problem (6) obtained by varying the parameters  $m$  and  $q_i$ . In spite of this, the mean-variance optimal portfolios are generally chosen by risk-averse investors who do not display prudence. Gamba and Rossi (1998), in fact, have shown that in a three fund separation context, prudent investors choose optimal portfolios with the same mean and greater skewness and variance of a minimum variance portfolio. Thus, the present analysis is substantially a generalization of the Markowitz one that permits one to determine the non-linearity aspect of risk. For this reason, we continue to refer to the *efficient frontier* (for a given category of investors) as the whole set of optimal choices (of that category of investors).

Moreover, as recently demonstrated by Athayde and Flôres (2004, 2005), when unlimited short sales are allowed and the risk measure is the variance, we can give an implicit analytical solution to the above problems using the tensorial notation for the higher moments. From these implicit solutions, we observe that the non-linearity of the above problems represents the biggest difference with the multi-parameter linear models proposed in the portfolio choice literature (see Ross (1976, 1978)). As a matter of fact, factor pricing models are generally well justified for large stock market aggregates. In this case, some general economic state (centered) variables  $Y_1, \dots, Y_{k-1}$  influence the pricing (see Chen, Roll, Ross (1986)). Recall that, most of the portfolio selection models depending on the first moments proposed in literature are  $k$ -fund separation models (see, among others, the three-moments based models proposed by Kraus and Litzenberg (1976), Ingersoll (1987), Simaan (1993), and Gamba and Rossi (1998)). Thus, they assume that each return follows the linear equation:

$$r_i = \mu_i + b_{i,1}Y_1 + \dots + b_{i,k-1}Y_{k-1} + \varepsilon_i \quad i=1, \dots, n, \quad (9)$$

where generally the zero mean vector  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)'$  is independent of  $Y_1, \dots, Y_{k-1}$  and the family of all convex combinations  $x'\varepsilon$  is a translation and scale invariant family

depending on a G-P translation invariant dispersion measure  $p(x'\varepsilon)$ . Then, when we require the rank condition<sup>9</sup> (see Ross (1978) and Ingersoll (1987)),  $k+1$  fund separation holds. Hence, if the riskless  $r_0$  is allowed, every risk-averse investor chooses a portfolio among the solution of the following constrained problem:

$$\begin{aligned} \min_x p(x'\varepsilon) \quad \text{subject to} \\ x'E(r) + (1 - \sum_i x_i)r_0 = m; \quad \sum_{i=1}^n x_i = 1; \\ x_i \geq 0; \quad i = 1, \dots, n; \quad x'b_{\cdot,j} = c_j; \quad j = 1, \dots, k-1 \end{aligned} \quad (10)$$

for some  $c_j$   $j=1, \dots, k-1$ , and an opportune mean  $m$ . Furthermore, if unlimited short selling is allowed and the vector  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)'$  is elliptical distributed with definite positive dispersion matrix  $V$  (see Owen and Rabinowitch (1983)), then all the solutions of (7) are given by:

$$\left(1 - \sum_{j=1}^k \lambda_j\right) r_0 + \lambda_1 \frac{r'V^{-1}(E(r) - r_0\mathbf{1})}{\mathbf{1}'V^{-1}(E(r) - r_0\mathbf{1})} + \sum_{j=1}^{k-1} \lambda_j \frac{r'V^{-1}b_{\cdot,j}}{\mathbf{1}'V^{-1}b_{\cdot,j}},$$

where  $\mathbf{1} = [1, 1, \dots, 1]'$  is a vector composed of ones and  $\lambda_i$   $i=1, \dots, k$  represent the weights in the  $k$  funds that together with the riskless asset span the efficient frontier (see Ortobelli (2001), Ortobelli *et al.* (2004)). Therefore, coherently with the classic arbitrage pricing theory the mean returns can be approximated by the linear pricing relation

$$E(r_i) = \mu_i = r_0 + \tilde{b}_{i,1}\delta_1 + \dots + \tilde{b}_{i,k}\delta_k$$

where  $\delta_j$  for  $j=1, \dots, k$ , are the risk premiums relative to the different factors. In particular, when we consider a three-fund separation model which depend on the first three moments, we obtain the so called Security Market Plane (SMP) (see, among others, Ingersoll (1987), Pressacco and Stucchi (2000), and Adcock *et al* (2005)). However, the approaches (6), (7), and (8) generalize the previous fund separation approach. As a matter of fact, if (9) is satisfied and all the portfolios are uniquely determined from the first  $k$

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<sup>9</sup> This further condition is required in order to avoid that the above model degenerates into a  $s$ -fund separation model with  $s < k+1$ .

moments, then the previous optimal solutions also can be parameterized with the first  $k$  moments. However, the converse is not necessarily true.

Let's assume that the portfolio returns  $x'r$  are uniquely determined by the mean and a G-P translation invariant positive and positively homogeneous risk measure  $p(1+x'r)$  and the skewness parameter  $\gamma_1(x'r)$ . Also suppose  $\bar{x}'r = \bar{x}'r(q_3)$  is the risky portfolio that maximizes the extended Sharpe measure for a fixed  $\gamma_1(x'r) = q_3$ . Then, for any  $\lambda \in (0,1)$ , an optimal portfolio with the same mean, skewness, and lower risk than  $z = \lambda r_0 + (1-\lambda)\bar{x}'r$  cannot exist because  $\frac{E(z-r_0)}{p(1+z)} = \frac{E(\bar{x}'r-r_0)}{p(1+\bar{x}'r)}$  and  $\gamma_1(z) = q_3$ . Thus, when unlimited short sales are allowed,<sup>10</sup> all the optimal choices are a convex combination of the riskless return and the solutions of the constrained problem

$$\begin{aligned} \max_x \frac{x'E(r)-r_0}{p(1+x'r)} \quad \text{subject to} \\ \sum_j x_j = 1; \quad \frac{E((x'r - E(x'r))^3)}{(x'Qx)^{3/2}} = q_3 \end{aligned} \tag{11}$$

varying the parameter  $q_3$ . However, we cannot guarantee that fund separation holds because the solutions of (11) are not generally spanned by two or more optimal portfolios. As typical example, we refer to the analysis by Athayde and Flôres (2004) and (2005) that assumes the variance as the risk measure.

As for the three-moments framework, we can easily extend the previous analysis to a context where all admissible portfolios are uniquely determined by a finite number of moments (parameters). Therefore, when returns present heavy tails and strong asymmetries, we cannot accept the  $k$  fund separation assumption. However, if we consider the presence of the riskless asset, then two-fund separation holds among portfolios with the same asymmetry parameters. On the other hand, the implementation of nonlinear portfolio selection models should be evaluated on the basis of the trade-off

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<sup>10</sup> When no short sales are allowed, we have to add the condition  $x_i \geq 0 \quad i=1, \dots, n$  at problem (11).

between costs and benefits. As a matter of fact, even the above moment analysis presents some non-trivial problems which are:

1) Estimates of higher moments tend to be quite unstable, thus rather large samples are needed in order to estimate higher moments with reasonable accuracy. In order to avoid this problem, Ortobelli et al (2003, 2004) proposed the use of other parameters to value skewness, kurtosis, and the asymptotic behavior of data.

2) We do not know how many parameters are necessary to identify the multi-parameter efficient frontier. However, this is a common problem on every multi-parameter analysis proposed in literature.

3) Even if the above optimization problems determine the whole class of the investor's optimal choices, those problems are computationally too complex to be solved for large portfolios, in particular when no short sales are allowed. Thus, we need to simplify the portfolio problems by reducing the number of parameters. When we simplify the optimization problem, for every risk measure we find only some among all optimal portfolios. Hence, we need to determine the risk measure that better characterizes and captures the investor's attitude.

## **5 AN EMPIRICAL COMPARISON AMONG THREE-PARAMETER EFFICIENT FRONTIERS**

Let us assume, for example, that the investors' choices depend on the mean, on the Pearson-Fisher skewness coefficient, and on a risk measure equivalent to a dispersion measure. Then, all risk-averse investors optimize their choices selecting the portfolios among the solutions of the following optimization problem:

$$\begin{aligned}
 & \min_x p(1 + x'r) \quad \text{subject to} \\
 & x'E(r) = m; \sum_i x_i = 1; x_i \geq 0 \quad i = 0, 1, \dots, n \\
 & \frac{E((x'r - E(x'r))^3)}{(x'Qx)^{3/2}} = q;
 \end{aligned} \tag{12}$$

for some mean  $m$  and skewness  $q$ . In this portfolio selection problem, we also consider the riskless asset that has weight  $x_0$ .

The questions we will try to answer are the following: Is the risk measure used to determine the optimal choices still important? If it is, which risk measure exhibits the best performance? What is the impact of skewness in the choices when we consider very asymmetric returns?

In order to answer to these questions, we consider the three risk measures discussed earlier: the MiniMax, the MAD, and standard deviation. These three measures are equivalent when portfolio distributions depend only on two parameters. In addition, when three parameters are sufficient to approximate investors' optimal choices, the optimal portfolio solutions of problem (12) with the three risk measures lead to the same efficient frontier (see Ortobelli (2001)).

### **5.1 Portfolio selection with and without the riskless return**

In the empirical comparison, we consider 804 observations of daily returns from 1/3/1995 to 1/30/1998 on 23 risky international indexes converted into U.S. dollars (USD) with the respective exchange rates.<sup>11</sup> In addition, we consider a fixed riskless asset of 6% annual rate. Solving the optimization problem (12) for different risk measures, we obtain Figure 2 on the mean-risk-skewness space. Here, we distinguish the efficient frontiers without the riskless asset (on the left) and with the riskless (on the right). Thus we can geometrically observe the linear effect obtained by adding a riskless asset to the admissible choices. As a matter of fact, when the riskless asset is allowed, all the optimal choices are approximately represented by a curved plane, even if no short sales are allowed. These efficient frontiers are composed of 5,000 optimal portfolios found by varying in problem (12) the mean  $m$  and the skewness  $q$  between the minimum (mean; skewness) and the maximum (mean; skewness).

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<sup>11</sup> We consider daily returns on DAX 30, DAX 100 Performance, CAC 40, FTSE all share, FTSE 100, FTSE actuaries 350, Reuters Commodities, Nikkei 225 Simple average, Nikkei 300 weighted stock average, Nikkei 300 simple stock average, Nikkei 500, Nikkei 225 stock average, Nikkei 300, Brent Crude Physical, Brent current month, Corn No2 Yellow cents, Coffee Brazilian, Dow Jones Futures1, Dow Jones Commodities, Dow Jones Industrials, Fuel Oil No2, and Goldman Sachs Commodity, S&P 500.

Generally, we cannot compare the three efficient frontiers because they are developed on different three-dimensional spaces. Thus, Figure 2 serves only to show that we could obtain different representations of the efficient frontiers when using different risk measures. Moreover, from Figure 2 we can also distinguish the optimal portfolios of risk averse, nonsatiable, prudent investors, i.e. the portfolios with the smallest risk and the highest mean and skewness. If three parameters are sufficient to describe the investor's optimal choices, then the optimal portfolio compositions obtained as solution of (12), corresponding to the three risk measures and fixed mean  $m$  and skewness  $q$ , must be equal. In this case, all three-parameter efficient frontiers represented on the same space must be equal. However, we have found that for any fixed mean  $m$  and skewness  $q$  the solution to the optimization problem (12) does not correspond to the same portfolio composition when we use different risk measures. From this difference, we deduce that three parameters are still insufficient to describe all the efficient portfolio choices.

Now, we introduce a comparison among mean-risk-skewness models from the perspective of some non-satiable risk-averse investors. We assume that several investors want to maximize their expected (increasing and concave) utility function. For every mean-risk-skewness efficient frontier, each investor will choose one of the 5,000 efficient portfolios. Thus, we obtain three optimal portfolios that maximize the expected utility on the three efficient frontiers. Comparing the three expected utility values, we can determine which efficient frontier better approximates the investor's optimal choice with that utility function. In particular, we assume that each investor has one among the following utility functions:

$$1) U(x'r) = \log(1 + x'r);$$

$$2) U(x'r) = \frac{(1 + x'r)^\alpha}{\alpha} \text{ with } \alpha = -5, -10, -15, -50; \quad 3) U(x'r) = -\exp(-k(1 + x'r))$$

with  $k = 8, 10, 11, 12, 13, 50$ .

In order to emphasize the differences in the optimal portfolio composition we denote by:

a)  $x_{best} = [x_{best,0}, x_{best,1}, \dots, x_{best,23}]$ , the optimal portfolio that realizes the maximum expected utility among the three different approaches;

b)  $x_{worst} = [x_{worst,0}, x_{worst,1}, \dots, x_{worst,23}]$  the optimal portfolio that realizes the lowest expected utility among the three approaches.

Then we consider the absolute difference between the two vectors of portfolio composition, i.e.  $\sum_{i=0}^{23} |x_{best,i} - x_{worst,i}|$ . This measure indicates in absolute terms how much change the portfolio considering different approaches. From a quick comparison of the estimated expected utility, major differences are not observed. However, the portfolio composition changes when we adopt distinct risk measures in the portfolio selection problems. That is, the portfolio composition is highly sensitive to small changes in the expected utility. For example, even if the difference between the highest and lowest optimal value of the exponential expected utility  $U(x'r) = -\exp(-50(1 + x'r))$  is of order  $10^{-22}$ , the corresponding optimal portfolio composition obtained in mean-standard deviation-skewness space is significantly different (about 37%) from that obtained in a mean-MiniMax-skewness space.

Table 4 summarizes the comparison among the three mean-risk-skewness approaches. In particular, we denote by "B" cases where the expected utility is the highest among the three models, "M" where the expected utility is the "medium value" among the three models, and "W" when the model presents the lowest expected utility. Table 4 shows that the optimal solutions are either on the mean-standard deviation-skewness frontier or on the mean-MiniMax-skewness frontier. Hence, investors with greater risk aversion obtain the best performance on the mean-standard deviation-skewness frontier, while less risk-averse investors maximize their expected utility on the mean-MiniMax-skewness efficient frontier.

Although we consider international indexes which lack substantial asymmetries, we observe some significant differences in the optimal portfolio compositions of investors with greater risk aversion. Instead, we do not observe very big differences in the optimal choices of less risk-averse investors. As a matter of fact, portfolio compositions of less

risk-averse investors present differences of order  $10^{-6}$  (that we approximate at 0%). On the other hand, even if the variance cannot be considered the unique indisputable risk measure that it has been characterized by in portfolio theory, this former empirical analysis confirms the good approximation of expected utility obtained in a mean, variance and skewness context (see Levy and Markowitz (1979) and Markowitz and van Dijk (2005)). Thus, we next investigate the effects of very asymmetric returns in portfolio choice.

## 5.2 An empirical comparison among portfolio selection models with derivative assets

As observed by Bookstaber and Clarke (1985), Mulvey and Ziemba (1999), and Iaquinta *et al.* (2003), the distribution of contingent claim returns present heavy tails and asymmetries. For this reason, it has more sense to propose a three-parameter portfolio selection comparison considering some contingent claim returns. Generally, we cannot easily obtain the historical observations of the same contingent claim. Thus, in order to capture the joint distributional behavior of asset derivatives, we need to approximate the historical observations of derivative returns.

In particular, mimicking the RiskMetrics' approximation of derivative's returns even for historical data (see, Longestaeey and Zangari (1996)), we can describe the returns of a European option with value  $V_t = V(P_t, K, \tau, r_0, \sigma)$  where  $P_t$  is the spot price of the underlying asset at time  $t$ ,  $K$ , the option's exercise price,  $\tau$ , the time to maturity of the option,  $r_0$ , the riskless rate, and  $\sigma$ , the standard deviation of the log return. Now, the value of the contingent claim can be written in terms of the Taylor approximation

$$V_{t+1} - V_t = \frac{1}{2} \Gamma (P_{t+1} - P_t)^2 + \Delta (P_{t+1} - P_t) + \Theta,$$

where we have used the Greeks  $\Gamma = \frac{\partial^2 V_t}{\partial P_t^2}$ ,  $\Delta = \frac{\partial V_t}{\partial P_t}$  and  $\Theta = \frac{\partial V_t}{\partial t}$ . Hence, the option

return  $R_t = \frac{V_{t+1} - V_t}{V_t}$  over the period  $[t, t+1]$  is approximated by the quadratic relation:

$$R_t = Ar_t^2 + Br_t + C, \tag{13}$$



where  $r_t = \frac{P_{t+1} - P_t}{P_t}$  is the return of the underlined asset, while  $A = \frac{P_t^2 \Gamma}{2V_t}$ ,  $B = \frac{P_t \Delta}{V_t}$  and

$C = \frac{\Theta}{V_t}$ . The main advantage of this approximation consists that we can analyze, describe

and evaluate the dependence structure of contingent claim portfolios. Moreover, as shown in Longerstaey and Zangari (1996), the relative errors of these approximations are reasonably low when options are not too close to the expiration date.

In this empirical analysis, we consider a subset of 10 of the risky international indexes used in the previous empirical analysis<sup>12</sup> and a fixed riskless asset of 6% annual rate. We approximated historical returns on six European calls and six European puts on the corresponding indexes. We assume that the options were purchased on 1/30/98 with a three months expiration. Thus, if we assume that non-linear approximation (13) holds with A, B and C fixed, then we can derive implicit approximations of a contingent claim return series considering i.i.d. observations of asset return  $r_t$ . Generally speaking, in order to obtain a better approximation of contingent claim returns, we follow the advise of RiskMetrics' empirical analysis.

Considering this portfolio composition, it is difficult to believe that three-fund separation holds and that the investors will all hold combination of no more than two mutual funds and the riskless asset. Then, we perform an analysis similar to the previous one based on the optimization problem (12), in order to value the impact and the differences of strongly asymmetric returns in the optimal investors' choices. Figure 3 shows the efficient frontiers we obtain by solving the optimization problem (12) for different risk measures. In this case, differences from the figures obtained previously are evident. In particular, the mean  $m$  and the skewness  $q$  of problem (12) vary in a larger interval and consequently we used 10,000 portfolios to approximate the efficient frontiers. Even in this case, we include a comparison among mean-risk-skewness models from the perspective of some non-satiable risk-averse investors. In addition, we want to

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<sup>12</sup> We consider daily returns from 1/3/1995 to 1/30/1998 on DAX 30, DAX 100 Performance, CAC 40, FTSE all share, FTSE 100, Nikkei 225 Simple average, , Nikkei 225 stock average, Dow Jones Industrials, Fuel Oil No2, S&P 500, and we consider puts and calls on DAX 30, CAC 40, FTSE 100, Nikkei 225, Dow Jones Industrials, and S&P 500. We convert all the returns into U.S. dollars with the respective exchange rates.

value the difference between the optimal choices obtained with the best of the three parameter models and the mean-variance optimal choices. Thus, we assume that each investor has one among the following utility functions:

$$1) U(x'r) = \log(1 + x'r);$$

$$2) U(x'r) = \frac{(1 + x'r)^\alpha}{\alpha} \text{ with } \alpha = -15, -25, -45, -55;$$

$$3) U(x'r) = -\exp(-k(1 + x'r)) \text{ with } k = 10, 20, 30, 55, 65, 75.$$

Then, as in the previous analysis, we compute the absolute difference between the two optimal portfolio that realizes the best and the worst performance among the three different approaches, i.e.  $\sum_i |x_{best,i} - x_{worst,i}|$ . In addition, we calculate the absolute difference between the portfolio that realizes the best performance  $x_{best}$  and the optimal portfolio that maximizes the expected utility on the mean variance efficient frontier that we point out with  $x_{MV} = [x_{MV,0}, x_{MV,1}, \dots, x_{MV,22}]$ . Thus, the measure  $\sum_i |x_{best,i} - x_{MV,i}|$  indicates in absolute terms how much the portfolio composition changes considering either a three parametric approach or the two parametric one.

Table 5 summarizes this empirical comparison. An analysis of the results substantially confirms the previous findings. In fact, the optimal solutions are either on the mean-standard deviation-skewness frontier or on the mean-MiniMax-skewness frontier. However, as we could expect, we observe much greater differences in the portfolio composition. Moreover, there exist significant differences between the mean-variance model and the three parametric ones. In particular, our empirical analysis suggests that:

- 1) The skewness parameter has an important impact in the portfolio choices when contingent claims are included in the optimization problem.

- 2) In the presence of returns with heavy tails and asymmetries, three parameters are still insufficient to evaluate the complexity of the portfolio choice problem,.
- 3) More risk-averse investors approximate their optimal choices on the mean-variance-skewness efficient frontier, while less risk-averse agents choose investments on the mean-MiniMax-skewness efficient.

## **6. CONCLUDING REMARKS**

In this paper we demonstrate that risk measures properties characterize the use of a risk measure. In particular, dispersion measures must be maximized at a fixed level of wealth under risk in order to obtain optimal portfolios for non-satiable investors. Thus, standard deviation, as with every dispersion measure, is not a proper risk measure. We observe that most of the risk measures proposed in the literature can be considered equivalent when the returns depend only on the mean and the risk. In this case, two-fund separation holds. However, when the return distributions present heavy tails and skewness, the returns cannot be generally characterized by linear models. In this case, we can only say that two-fund separation holds among portfolios with the same asymmetry parameters when the riskless asset is present.

Finally, a preliminary empirical analysis shows that there are still motivations to analyze the impact of different risk measures and of skewness in portfolio theory and that three parameters are still insufficient to evaluate the complexity of a portfolio choice problem, in particular when we consider contingent claim returns.

Further analysis, comparison, and discussion are still necessary to decide which risk measure gives the best performance. Probably, for this purpose it is better to compare only mean-risk models because the impact that a risk measure has in portfolio choice is much more evident. On the other hand, many other aspects of distributional behavior of asset returns should be considered. As a matter of fact, several studies on the empirical behavior of returns have reported evidence that conditional first and second moments of stock returns are time varying and potentially persistent, especially when returns are measured over long horizons. Therefore, it is not the unconditional return distribution which is of interest but the conditional distribution which is conditioned on information

contained in past return data, or a more general information set. In addition, the assumption of conditional homoskedasticity is often violated in financial data where we often observe volatility clustering and the class of auto-regressive (moving average) with auto-regressive conditional heteroskedastic AR(MA)-GARCH models is a natural candidate for conditioning on the past of return series. In this context the complexity of portfolio selection problems could grow enormously (see, among others, Tokat et al (2003), Bertocchi et al (2005)). However, in some cases, it can be reduced by either considering the asymptotic behavior of asset returns (see for example Rachev and Mittnik (2000) and Ortobelli et al. (2003, 2004) and the reference therein) or considering alternative equivalent optimization problems that reduce the computational complexity. (see Rachev et al (2004, 2005), Biglova et al. (2004)).

## REFERENCES

1. C. Adcock, C., Jurczenko, E. and B., Maillet 2005. *Multi-moment capital asset pricing models and related topics*. Berlin: Springer Verlag.
2. Artzner, P. Delbaen, F., Heath, J-M., and D., Eber 2000. Coherent measures of risk, *Mathematical Finance* **9**, 203-228.
3. Artzner, P. Delbaen, F., Heath, J-M., Eber, D., and H. Ku 2003. Coherent multiperiod measures of risk.
4. Athayde, G. M. and R. G., Flôres Jr. 2004. Finding a maximum skewness portfolio – a general solution to three-moments portfolio choice. *Journal of Economic Dynamics and Control* **28**; 1335-52.
5. Athayde, G. M. and R. G., Flôres Jr. 2005. On certain geometric aspects of portfolio optimisation with higher moments, in (Adcock, Jurczenko, and Maillet, eds.), *Multi-Moment Capital Asset Pricing Models and Related Topics*. Berlin: Springer Verlag.
6. Balzer, L.A., 2001. Investment risk: a unified approach to upside and downside returns, in (Sortino F. and Satchell eds.) *Managing downside risk in financial markets: theory practice and implementation*, Butterworth-Heinemann, Oxford, 103-156.
7. Bertocchi M., Giacometti R., Ortobelli S. and S., Rachev 2005 The impact of different distributional hypothesis on returns in asset allocation to appear in *Finance Letters*.
8. Biglova A., Ortobelli S., Rachev S., and S., Stoyanov 2004. different approaches to risk estimation in portfolio theory, *Journal of Portfolio Management* **31**, 103-112.
9. Bookstaber, R. and R., Clarke 1985. Problems in evaluating the performance of portfolios with options, *Financial Analysts Journal*, Jan-Feb.48-62.
10. Chen, N., R. Roll and S., Ross 1986. Economic forces and the stock market, *Journal of Business* **59**, 383-403.
11. Gaivoronski A., and G., Pflug 2001. Value at risk in portfolio optimization: properties and computational approach, *Technical Report*, University of Vienna.

12. Gamba, A. and F., Rossi 1998. Mean variance skewness analysis in portfolio choice and capital markets, *Journal of the Italian Operational Research Society* **85**, 5-46.
13. Giacometti R., and S., Ortobelli 2004. Risk measures for asset allocation models in (Szegö G. ed) *Risk measures for the 21<sup>st</sup> century*, Wiley & Son, Chichester, 69-87.
14. Horvath, P.A. and R.C., Scott 1980. On the direction of preference for moments of higher order than variance, *Journal of Finance* **35**, 915-919.
15. Iaquinta G., Lamantia F., Massabò I. and S., Ortobelli 2003. A semi-parametric approach to value the risk of asset derivative portfolios, *Technical Report 15*, Università di Bergamo
16. Ingersoll, J. Jr. 1987. Theory of financial decision making, Totowa: Rowman & Littlefield.
17. Kimball M.S. 1990. Precautionary savings in the small and in the large, *Econometrica* **58**, 53-73.
18. Konno, H. and H., Yamazaki 1991. Mean-absolute deviation portfolio optimization model and its application to Tokyo stock market, *Management Science* **37**, 519-531.
19. Kraus, A. and R., Litzenberger 1976. Skewness preference and the valuation of risk assets, *Journal of Finance* **31**, 1085-1100.
20. Levy, H. and H., Markowitz 1979. Approximating expected utility by a function of mean and variance, *American Economic Review* **69**, 308-317
21. Levy, H., 1992. Stochastic dominance and expected utility: survey and analysis, *Management Science* **38**, 555-593.
22. Longestaey, J. and P., Zangari 1996. *RiskMetrics -technical document* J.P. Morgan, Fourth edition, New York.
23. Markowitz, H. 1952. Portfolio selection, *Journal of Finance* **7**, 77-91.
24. Markowitz, H. 1959. Portfolio selection; efficient diversification of investment, New

York: Wiley.

25. Markowitz, H. and E., van Dijk 2005. Risk return analysis, to appear in the *Handbook of Asset and Liability Management*, North Holland Handbooks of Finance (Series Editor W. T. Ziemba).
26. Mulvey J., and W. Ziemba 1999. Asset and liability management systems for long term investors: discussion of the issues, *Worldwide Asset and Liability Modeling*, Cambridge University Press.
27. Ogryczak, W. and A., Ruszczyński 1999. From stochastic dominance to mean-risk models: semideviations as risk measures, *European Journal of Operational Research* **116**, 33-50.
28. Ortobelli S., 2001. The classification of parametric choices under uncertainty: analysis of the portfolio choice problem, *Theory and Decision* **51**, 297-327.
29. Ortobelli, S., Huber I., Rachev S. and E., Schwartz 2003. Portfolio choice theory with non-Gaussian distributed returns, in *Handbook of Heavy Tailed Distributions in Finance*, 547-594, North Holland Handbooks of Finance Series Editor W. T. Ziemba.
30. Ortobelli S., Rachev S., Huber I. and A., Biglova 2004. Optimal portfolio selection and Risk management: A comparison between the stable Paretian approach and the Gaussian one in Chapter 6, *Handbook of Computational and Numerical Methods in Finance*, 197-252.
31. Owen, J. and R., Rabinovitch 1983. On the class of elliptical distributions and their applications to the theory of portfolio choice, *Journal of Finance* **38**, 745-752.
32. Pressacco, F. and P., Stucchi 2000. Linearity properties of a three moments portfolio model, *Decisions in Economic and Finance* **23**, 133-150
33. Rachev, S. and S., Mittnik 2000. Stable Paretian model in finance, Wiley & Sons, Chichester.
34. Rachev S.T., Ortobelli S. and E., Schwartz 2004. The problem of optimal asset allocation with stable distributed returns, in Volume 238 (Alan C. Krinik, Randall J. Swift eds.) *Stochastic Processes and Functional Analysis* Marcel Dekker Inc., 295-347.

35. Rachev S. T., Ortobelli S., Stoyanov S., Fabozzi F. and A., Biglova 2005 Desirable properties of an ideal risk measure in portfolio theory, to appear as *Technical Report*, University of Karlsruhe (International Summer School in “Risk Measurement and Management” Rome, June 2005).
36. Rockafeller R.T., Uryasev S., and M., Zabarankin 2003. Portfolio analysis with general deviation measures, *Research Report*, 2003-8, University of Florida.
37. Ross, S. 1976. The arbitrage theory of capital asset pricing, *Journal of Economic Theory* **13**, 341-360.
38. Ross, S. 1978. Mutual fund separation in financial theory: The separating distributions, *Journal of Economic Theory* **17**, 254-286.
39. Rothschild, M. and J., Stiglitz 1970. Increasing risk: I. definition, *Journal of Economic Theory* **2**, 225-243.
40. Simaan, Y. 1993. Portfolio selection and asset pricing: Three parameter framework, *Management Science* **5**, 568-577.
41. Speranza, M. G. 1993. Linear programming models for portfolio optimization, *Finance* **14**, 107-123.
42. Tobin, J. 1958. Liquidity preference as behavior toward risk, *Review of Economic Studies* **25**, 65-86.
43. Tokat, Y., S. T. Rachev, and E., Schwartz 2003. The stable non-Gaussian asset allocation: A comparison with the classical Gaussian approach, *Journal of Economic Dynamics & Control* **27**, 937-969
44. Young, M.R. 1998. A minimax portfolio selection rule with linear programming solution, *Management Science* **44**, 673-683.



<b>Properties</b>	<b>Proper risk Measures</b>	<b>Uncertainty Measures</b>
Consistency with respect to first order stochastic dominance due to additive shifts	yes	no
Consistency with respect to first order stochastic dominance	yes	no
Monotony	yes	no
Consistency with respect to second order stochastic dominance	yes	no
Consistency with respect to Rothschild-Stiglitz stochastic order	no	yes
Positively homogeneous	yes	yes
Convexity	yes	yes
Subadditive	yes	yes
Positive	no	yes
Gaivoronsky-Pflug translation invariant	no	yes
Translation invariant	yes	no
Functional translation invariant	yes	yes
Coherent risk measure	yes	no
Deviation measure	no	yes
Expectation-bounded risk measure	yes	no

**Table 1.** Properties of uncertainty measures and proper risk measures.

<b>RISK MEASURES</b>	<b>PROPERTIES</b>
$VaR_\alpha(W_x)$ Value at Risk $-\inf\{z   \Pr(W_x \leq z) > \alpha\}$	Safety risk measure that is monotone; consistent with FSD stochastic order; positively homogeneous; and translation invariant.
$CVaR_\alpha(W_x)$ Conditional Value at Risk $E(-W_x   -W_x \geq VaR_\alpha(W_x))$	Safety risk measure that is monotone; consistent with FSD, SSD, R-S stochastic orders; positively homogeneous; convex; sub-additive; linearizable; coherent; translation invariant and expectation-bounded.
$MM(W_x)$ MiniMax $-\sup\{c \in R   \Pr(W_x \leq c) = 0\}$	Safety risk measure that is monotone; consistent with FSD, SSD, R-S stochastic orders; positively homogeneous; convex; sub-additive; linearizable; coherent; translation invariant and expectation-bounded.
Safety First $\Pr(W_x \leq \lambda)$ Lower Partial Moment $\sqrt[q]{E((W_x - Y)_-^q)}$ ,	Safety risk measure that is consistent with FSD stochastic order and monotone.
where $q \geq 1$ is the power index, $Y$ is the target wealth. $CVaR_{\alpha,q}(W_x)$ Power CVaR $E( W_x ^q   -W_x \geq VaR_\alpha(W_x))$	Safety risk measure that is monotone; consistent with FSD, SSD, R-S stochastic orders; convex; and sub-additive.
where $q \geq 1$ is the power index.	

**Table 2** Properties of safety risk measures

## RISK MEASURES

Standard Deviation  

$$\sqrt{E((W_x - E(W_x))^2)}$$

MAD

$$E(|W_x - E(W_x)|).$$

Mean-absolute moment

$$\left(E(|W_x - E(W_x)|^q)\right)^{1/q},$$

where  $q \geq 1$ .

Gini's mean difference

$$E(|W_x - Y|),$$

where  $Y$  points out an i.i.d. copy of wealth  $W_x$ .

Exponential entropy (only for wealth that admit a density distribution)

$$e^{-E(\log f_{W_x}(t))}$$

where  $f_{W_x}(t)$  is the density of wealth  $W_x$ .

Colog of  $W_x$

$$E(W_x \log(W_x)) - E(W_x)E(\log(W_x)).$$

## PROPERTIES

Deviation measure that is positive; consistent w.r.t. R-S stochastic order; positively homogeneous; convex; sub-additive; and G-P translation invariant.

Deviation measure that is positive; consistent w.r.t. R-S stochastic order; positively homogeneous; convex; sub-additive; linearizable; and G-P translation invariant.

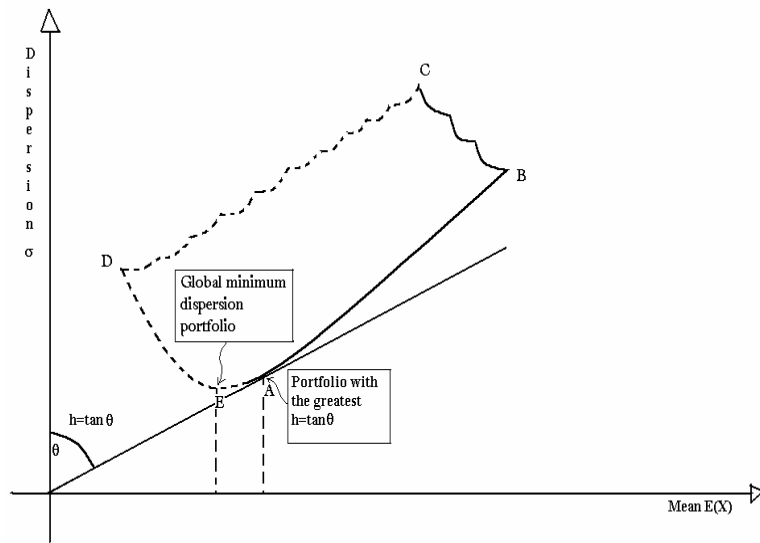
Deviation measure that is positive; consistent w.r.t. R-S stochastic order; convex; positively homogeneous; G-P translation invariant and sub-additive.

Deviation measure that is positive; consistent w.r.t. R-S stochastic order; positively homogeneous; convex; sub-additive; linearizable; and G-P translation invariant.

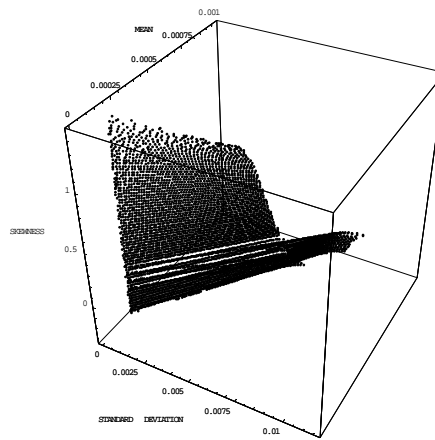
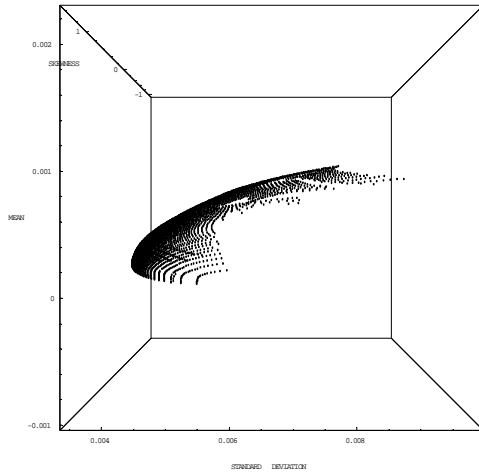
Deviation measure that is positive; consistent w.r.t. R-S stochastic order; positively homogeneous; convex; sub-additive; and G-P translation invariant.

Risk measure that is positive; consistent w.r.t. FSD due to additive shifts and R-S stochastic order; positively homogeneous; convex and sub-additive.

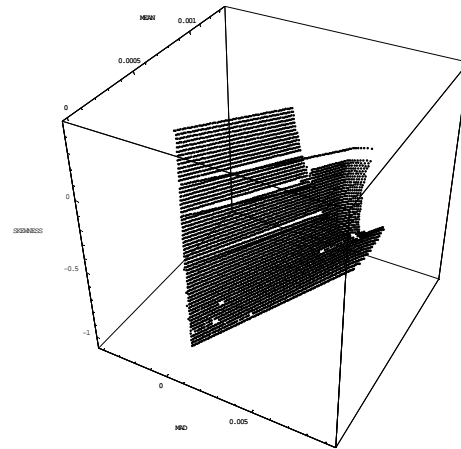
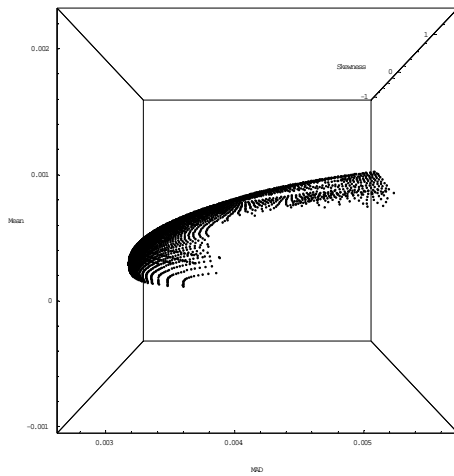
**Table 3** Properties of Dispersion Risk Measures



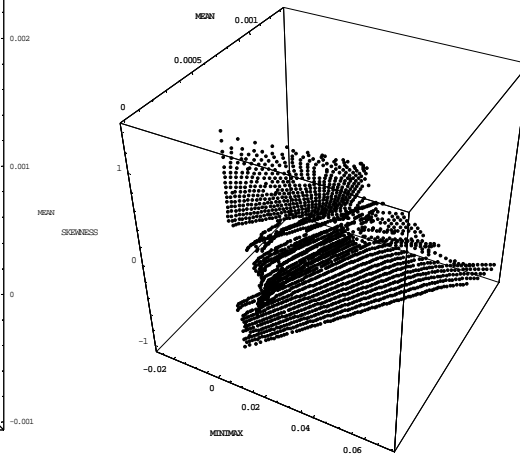
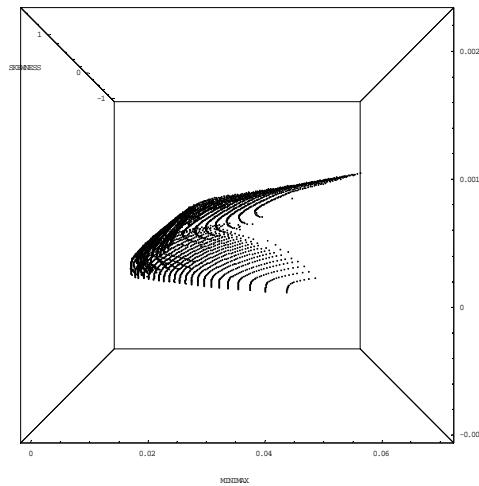
**Figure 1.** \_\_\_ Efficient portfolios for non-satiable investors; - - - Non-optimal portfolios.



Mean-Standard Deviation-Skewness optimal choices without and with the riskless.

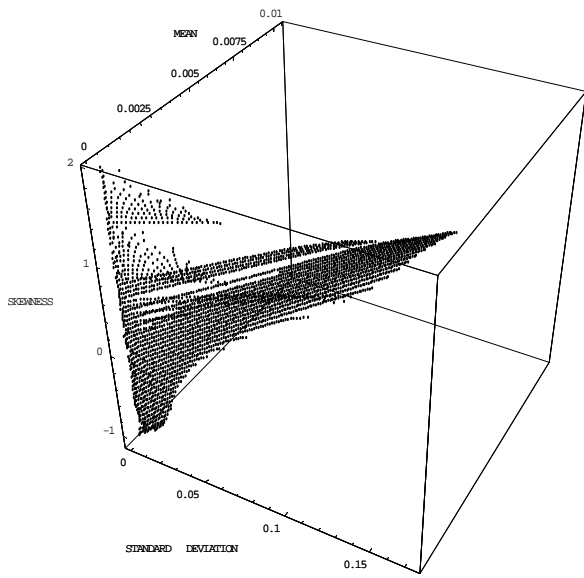


Mean-MAD-Skewness optimal choices without and with the riskless.

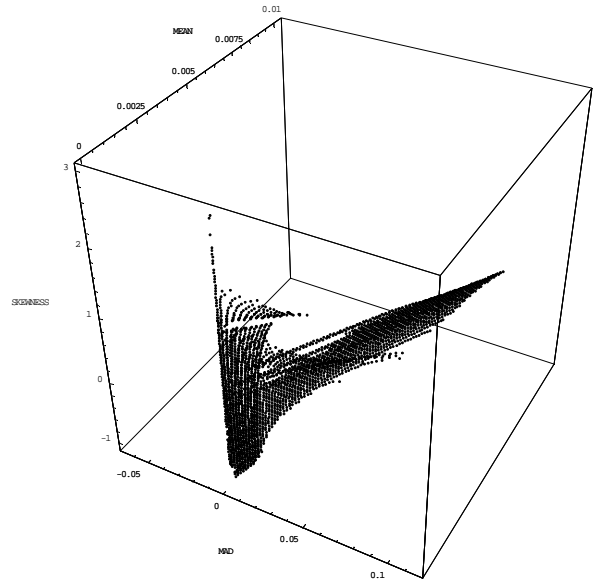


Mean-Minimax-Skewness optimal choices without and with the riskless..

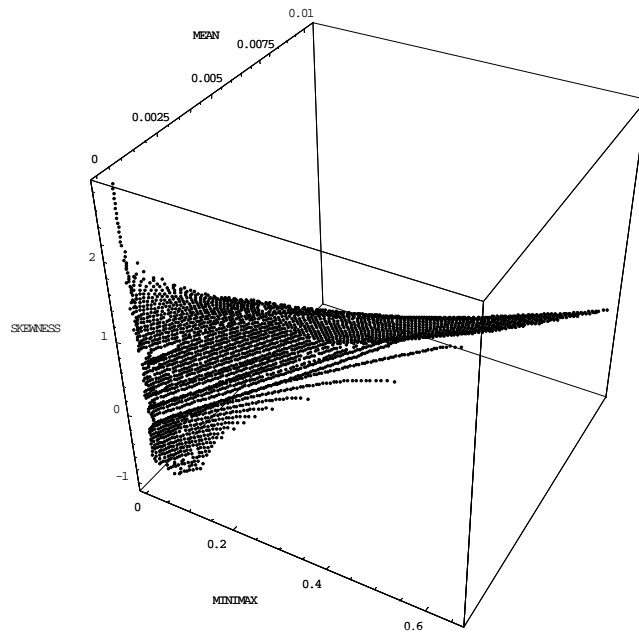
**Figure 2.** Three-parameter efficient frontiers for risk-averse investors (without and with the riskless asset).



Mean-Standard deviation-Skewness optimal choices.



Mean-MAD-Skewness optimal choices



Mean-Minimax-Skewness optimal choices.

**Figure 3.** Three-parameter efficient frontiers for risk-averse investors considering portfolios of derivatives.

Expected Utility	Mean – Standard Deviation- Skewness	Mean-MAD- Skewness	Mean- Minimax- Skewness	Difference between portfolio composition
				$\sum_{i=0}^{23}  x_{best,i} - x_{worst,i} $
$E(\log(1+x \cdot r))$	M	W	B	0%
$\frac{-1}{5} E\left((1+r_p)^{-5}\right)$	M	W	B	0%
$\frac{-1}{10} E\left((1+r_p)^{-10}\right)$	M	W	B	8.50%
$\frac{-1}{15} E\left((1+r_p)^{-15}\right)$	B	W	M	18.10%
$\frac{-1}{50} E\left((1+r_p)^{-50}\right)$	B	M	W	38.10%
$-E(\exp(-8(1+r_p)))$	M	W	B	0%
$-E(\exp(-10(1+r_p)))$	W	M	B	4.30%
$-E(\exp(-11(1+r_p)))$	W	M	B	6.90%
$-E(\exp(-12(1+r_p)))$	B	W	M	6.40%
$-E(\exp(-13(1+r_p)))$	B	W	M	9.10%
$-E(\exp(-50(1+r_p)))$	B	M	W	37.30%

**Table 4** Attitude to risk of some investors on three parametric efficient frontiers and analysis of the models' performance. We maximize the expected utility on the efficient frontiers considering daily returns from 1/3/1995 to 1/30/1998 on 23 risky international indexes and a fixed riskless return. We write "B" when the expected utility is the highest among the three models, we write "M" when the expected utility is the "medium value" among the three models and we write "W" when the model presents the lowest expected utility.

Expected Utility	Mean-Standard-Deviation-Skewness	Mean-MAD-Skewness	Mean-MiniMax-Skewness	Difference between portfolio composition	Difference between portfolio composition
				$\sum_{i=0}^{22}  x_{best,i} - x_{worst,i} $	$\sum_{i=0}^{22}  x_{best,i} - x_{MV,i} $
$E(\log(1+x'r))$	W	M	B	6.21%	81.33%
$\frac{-1}{15}E\left((1+r_p)^{-15}\right)$	W	M	B	8.02%	45.16%
$\frac{-1}{25}E\left((1+r_p)^{-25}\right)$	M	W	B	14.52%	25.10%
$\frac{-1}{45}E\left((1+r_p)^{-45}\right)$	B	W	M	27.73%	38.37%
$\frac{-1}{55}E\left((1+r_p)^{-55}\right)$	B	M	W	48.17%	52.09%
$-E(\exp(-10(1+r_p)))$	W	M	B	6.59%	86.33%
$-E(\exp(-20(1+r_p)))$	W	M	B	10.52%	50.09%
$-E(\exp(-30(1+r_p)))$	M	W	B	17.49%	26.72%
$-E(\exp(-55(1+r_p)))$	B	W	M	26.44%	28.06%
$-E(\exp(-65(1+r_p)))$	B	W	M	35.94%	43.22%
$-E(\exp(-75(1+r_p)))$	B	M	W	42.31%	46.33%

**Table 5** Attitude to risk of some investors on three parametric efficient frontiers and analysis of the models' performance when we consider portfolios of asset derivatives. We maximize the expected utility on the efficient frontiers considering a fixed riskless return, the approximated historical daily returns of 12 asset derivatives and daily returns on 10 risky international indexes. We write "B" when the expected utility is the highest among the three models, we write "M" when the expected utility is the "medium value" among the three models and we write "W" when the model presents the lowest expected utility.