The Impact of Different Distributional Hypothesis on Returns in Asset Allocation

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Abstract

This paper discusses and analyzes some portfolio allocation problems with autoregressive processes. Firstly, we examine the distributional behavior of some international indexes. Then, we propose an AR(1)-GARCH(1,1) model to describe the evolution of the portfolio returns over time. In particular, we assume that investors wish to maximize some expected utility functionals of final wealth and we evaluate the impact of different distributional hypotheses on investor's choices. Finally, we describe the properties of some optimal choices.

Key words: Portfolio selection, Stable distributions, generalized Student's t_3, AR(1)-GARCH(1,1) models.
JEL Classification: G11, G12, G14.

1. INTRODUCTION

Many of the concepts developed in theoretical and empirical finance over the past decades rest upon the assumption that asset returns follow a normal distribution. In particular, this distributional assumption was applied to the one period strategic allocation problem in order to justify the mean variance analysis. Such an approach, called myopic, was partially motivated by Samuelson (1969) and Merton (1969) who proved that when we assume a perfect market, the investors with constant relative risk aversion maintain constant their portfolio compositions independently of the temporal horizon. On the other hand, since the space of feasible consumption bundles is generally a linear space (as Ross (1978), Cox and Leland (1982), Rubinstein (1976) and many others have emphasized), the original dynamic problem can be replaced with an equivalent one-period problem which has appropriate terminal state prices, if consumption takes place at the end. More generally, if preferences are time separable and if we treat consumption at each date separately, the analysis is unchanged. However, the myopic approach presents several drawbacks. First of all, it does not consider properly the return predictability empirically observed. Secondly, this approach cannot be used for long term approaches when transaction costs are allowed.

In order to evaluate how the investors' choices change when the return predictability is modeled, Wilkie (1995), Hodrick (1992), Boender (1997) have proposed several autoregressive models to describe the evolution of the asset returns over time. However, most of these models assume that returns or their stochastic innovations are Gaussian distributed. This distributional assumption is against the empirical evidence. The excess kurtosis,

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found in Mandelbrot's (1963) and Fama's (1965) investigations, led them to reject the normal assumption and to propose the stable Paretian distribution as a statistical model for asset returns. In addition, many other empirical analyses (see Rachev and Mittnik (2000) and the references therein) have shown that residuals of autoregressive models applied to daily financial series are not Gaussian distributed.

In this paper we relax the Gaussian distributional hypothesis on the stochastic innovations of returns and we evaluate the impact of different distributional hypotheses on multiperiod portfolio choices. Our analysis is the starting point for further developments of the most recent and used multiperiod stochastic portfolio choice models (see Dupacovà (1999), Ziemba and Mulvey (1999), Kouwenberg, Zenios (2001), Dempster et al (2003), Zenios and Ziemba (2004)). In particular, we assume that every portfolio of risky returns evolves as an AR(1)-GARCH(1,1) model. Therefore, we consider the autoregressive component of the returns and the heteroschedasticity of their volatility. In addition, we distinguish three different distributional hypotheses for the standardized residuals and we assume that they could be either Gaussian or t_3 or α- stable distributed. We generate several scenarios for each portfolio and distributional hypothesis in order to compute the expected utility of the final wealth. Specifically, we analyze three investment allocation problems. Using the Simulated Annealing algorithm we choose the optimal composition among a set of admissible portfolios.

From a financial point of view, we typically propose and solve a fixed-mix portfolio choice problem following Boender (1997) and Tokat, Rachev and Schwartz (2003) analysis but differently we compare the allocation obtained with the Gaussian, the stable non-Gaussian and the Student distributional assumption for the residuals of AR(1)-GARCH(1,1) approximation of risky portfolios. To select the portfolio we consider the daily observations of the stock indexes S&P500, DAX30 and CAC40.

2. MODELING UNCERTAINTY IN RETURNS

As it is well known in literature and suggested by many authors, see Tokat and Schwartz (2002), financial modeling often requires information about the past movements of financial variables as prices or returns. Let us suppose that our portfolio horizon is T and the number of risky assets involved in the portfolio is n. Let $P_{i,t}$ be the price of the asset i at time t, we define the return $R_{i,t}$ of the asset i at time t as

$$R_{i,t} = \log \frac{P_{i,t+1}}{P_{i,t}} \quad i=1,\dots,n; \quad t=1,\dots,T. \tag{1}$$

The contribution of each asset i in the portfolio is represented by the quantity x_i. The portfolio return, $R_{p,t}$ at time t, is defined as

$$R_{p,t} = \mathbf{x}'\mathbf{R}_t \tag{2}$$

where $\mathbf{R}_t = [R_{1,t},\dots,R_{n,t}]'$ and $\mathbf{x}' = [x_1,\dots,x_n]$. There are many alternative methods to describe the uncertainty in asset returns, we decided to take into consideration those where the future return distribution is conditioned by information sets based on past returns and the time series exhibit some kind of volatility cluster.

2.1. *AR-GARCH modeling*

Here we refer to the generalized autoregressive conditional heteroskedastic model (GARCH) of Bollerslev (1986) combined with an autoregressive model (AR). The choice of an AR(1)-GARCH(1,1) for our portfolio arises from the analysis of the time series of the asset returns in our portfolio. A detailed description will be given in the next Section. Using an AR(1)-GARCH(1,1) to shape the behavior of $R_{p,t}$, we derive the following model

$$R_{p,t} = \alpha_0 + \alpha_1 R_{p,t-1} + \varepsilon_t \tag{3}$$

$$\sigma_t^2 = \gamma_0 + \gamma_1 \varepsilon_{t-1}^2 + \gamma_2 \sigma_{t-1}^2 \tag{4}$$

Observe that all the parameters in equations (3), (4) have to be estimated using the observed data available on the market. In the classical model the innovations ε_t are normally distributed with zero mean and variance σ_t^2, i.e. $\varepsilon_t \mid I_{t-1} \approx N\left(0,\sigma_t^2\right)$, where I_{t-1} is the information set at time t-1. However, many studies have shown that the residuals from GARCH models are strongly inconsistent with normality and exhibit heavy tails and skewness. Thus, alternatively to the Gaussian hypothesis, we assume α-stable and t_3 distributions for standardized residuals. The standard procedure to carry out the estimation is the well-known maximum-likelihood methodology and it consists in maximizing the logarithm of the conditional likelihood function built on the basis of residuals distributions whose properties are recalled here in the following.

2.1.1. *The α - stable distribution*

The α-stable distributions describe a general class of distribution functions. A random variable X is α- stable distributed if it has a domain of attraction, that is there exists a sequence of i.i.d. random variables $\left\{Y_i\right\}_{i\in N}$, a sequence of positive real values $\left\{d_i\right\}_{i\in N}$ and a sequence of real values $\left\{a_i\right\}_{i\in N}$ such that, as $n\to+\infty$

$$\frac{1}{d_n}\sum_{i=1}^{n} Y_i + a_n \xrightarrow{d} X \tag{5}$$

where "$\xrightarrow{d}$" points out the convergence in the distribution. Thus, the α-stable random variables describe a general class of distributions including the leptokurtic and asymmetric ones. The α-stable distribution is identified by four parameters: the index of stability $\alpha\in(0,2]$ which is the parameter of the kurtosis, the skewness parameter $\beta\in[-1,1]$; $\delta\in\Re$ and $\gamma\in\Re^+$ which are, respectively, the location and the dispersion parameter. If X is a random variable whose distribution is α-stable, we use the following notation to underline the parameter dependence

$$X \overset{d}{=} S_\alpha(\gamma,\beta,\delta)\,. \tag{6}$$

The stable distribution is Normal, when α=2 and it is leptokurtotic when α<2. A positive skewness (β>0) identifies distributions with right fat tails, while a negative skewness (β<0) typically characterizes distributions with left fat tails. Therefore, the stable density functions synthesize the distributional forms empirically observed in the real data. The estimation of stable distribution parameters takes place by maximizing the likelihood function (see Rachev and Mittnik (2000)).

2.1.2. *t_3 distribution*

Alternatively to the stable distributed standardized residuals, we consider the so-called t_3 distribution (see Mittnik and Paolella (2000)). The t_3 distribution contains three parameters and it is characterized by the following density

$$f\left(z;d,\theta,v\right) = K\cdot\begin{cases}\left(1+\dfrac{\left(-z\theta\right)^d}{v}\right)^{-\left(v+\frac{1}{d}\right)} & if\ \ z<0\\[2ex] \left(1+\dfrac{\left(z/\theta\right)^d}{v}\right)^{-\left(v+\frac{1}{d}\right)} & if\ \ z\geq 0\end{cases} \tag{7}$$

where $K^{-1}=(\theta+\theta^{-1})d^{-1}v^{1/d}B\left(d^{-1},v\right)$ and $B(a,b)=\left(\Gamma\left(a\right)\Gamma\left(b\right)\right)\Big/\left(\Gamma\left(a+b\right)\right)$ denotes the beta function. It has been shown that the distribution is unimodal, the parameter $v>0$ represents the degrees of freedom that together with $d>0$ measures the heaviness of the distribution tails and $\theta>0$ represents the skewness of the distribution.

The t_3 distribution is a generalization of the generalized exponential as well as the Student's t, Laplace, Cauchy and the Gaussian as ν tends to infinity. For $d=2$, $\theta=1$ we obtain the centered Student's-t distribution with ν degrees of freedom and for opportune parameters we could obtain the other distributions (see Mittnik and Paolella (2000)).

The t_3 distributions present fatter tails than the Gaussian. In particular, vd provides the maximal existing moment of the unobservable innovations process. Thus, if the estimated product vd is less than 2, we implicitly assume that the t_3 distributed residuals have infinite variance (see Mittnik and Paolella (2000)).

A positive skewness ($\theta>1$) identifies distributions with right fat tails, while a negative skewness ($\theta<1$) typically characterizes distributions with left fat tails.

3. EMPIRICAL EVIDENCE

In this section we employ different criteria for comparing candidate distributional assumptions. In our comparison we use daily data on three index-daily returns: DAX 30, CAC 40, S&P 500 quoted on the stock market from January 1994 to January 1998.

In order to test the normality of asset returns we consider their mean, standard deviation σ, the Pearson skewness and kurtosis parameters and other four different risk measures of the risky indexes used in our portfolio analysis: $VaR_{99\%}$, $VaR_{95\%}$, $CVaR_{99\%}$, $CVaR_{95\%}$ (see Table I).

	Mean	**Sigma**	**Skewness**	**Kurtosis**	**VaR$_{99\%}$**	**VaR$_{95\%}$**	**CVaR$_{99\%}$**	**CVaR$_{95\%}$**
DAX 30	0.0746	1.1339	-0.5353	7.6844	-3.1902	-1.9542	-4.0757	-2.745
CAC 40	0.0481	1.1048	0.0887	5.308	-2.7602	-1.7976	-3.3539	-2.531
S&P 500	0.0945	0.8326	-0.7287	12.0318	-2.2179	-1.1776	-3.2157	-3.004

Table I

This table summarizes parameter estimates of three international indexes. In particular, we consider: the mean, the standard deviation, the skewness, the kurtosis, the VaR$_{99\%}$, the VaR$_{95\%}$, the CVaR$_{99\%}$, the CVaR$_{95\%}$. All parameters are computed on series of daily returns between January 1994 and January 1998 for a total of 948 observations.

From the analysis of these distributional parameters we observe that all the indexes present kurtosis and skewness far from the normal values. Moreover, even if S&P500 dominates the other indexes in a mean-variance framework, this index presents the greatest kurtosis and conditional value at risk at 95% confidence level.

A first empirical analysis on the historical return series indicates a considerable deviation from the normality (see Bertocchi et al. (2004)). The maximum likelihood estimation of α-stable and t_3 parameters of return series shows that the index of stability is always below two and that vd is not very high, then the return series present heavy tails. Moreover, the stable index of skewness β is always different from zero and the series presenting skewness do not present the strong asymmetry $\theta\approx 1$. In addition, we employ two tests which are based on the whole empirical distribution. Other goodness-of-fit measures can also be used, see Mittnik and Paolella (2000) and Rachev and Mittnik (2000). In particular, the distribution of index returns has been analyzed using the Kolmogorov-Smirnoff (K-S) test

$$KS=\sup_{x\in R}\left|F_E(x)-F(x)\right| \tag{8}$$

and the Anderson-Darling (A-D) statistic (see Anderson-Darling (1952))

$$AD = \max_{x \in R} \frac{\left| F_E(x) - F(x) \right|}{\sqrt{F(x)\left(1 - F(x)\right)}} \tag{9}$$

where F_E is the empirical cumulative distribution and F the chosen cumulative model distribution.

The A-D statistic weighs discrepancies appropriately across the whole support of the distribution and it is particularly important if one is interested in determining tails of return distributions. The K-S test and the A-D statistic (see Bertocchi et al. (2004)) show that the t_3 distribution presents optimal performance in approximating the empirical distribution. In particular as a consequence of A-D tests we observe that the Stable distribution presents a better approximation to the tails among the three distributions proposed. Therefore, the proposed tests agree upon in rejecting the hypothesis of normality.

Next we analyze conditional homoskedastic and heteroskedastic distributions. Thus, we could compare the empirical cumulative distribution F_E of several standardized portfolio residuals with either a simulated Gaussian or a non-Gaussian distribution F.

3.1. *Autoregressive dependency*

The study and the analysis conducted on the empirical behavior of data (see, among others, French, Schwert and Stambaugh (1987)) have reported evidence that conditional first and second moments of stock returns are time varying and potentially persistent especially when returns are measured over long horizons.

In order to test the ARMA and GARCH dependencies we followed the standard Box-Jenkins identification techniques. More specifically, we inspected the sample autocorrelation functions (SACF) and the sample partial autocorrelation functions (PACF) of the return series r_t to test the ARMA dependencies and of the return series r_t^2 to describe the GARCH dependencies. The exponentially decaying SACF and the single large spike in PACF corresponding to our three historical series suggest the appropriateness of an AR(1) structure while there is not strong evidence of the moving average MA(1) component (see Table II). In addition, Table II shows that the standard Box-Jenkins methodology would suggest the need for a mixed model. That is, we find that GARCH(1,1) is adequate in capturing the correlation structure of these squared series.

		AR(1)		**GARCH(1)**	**ARCH(1)**
	α_0	α_1	γ_0	γ_1	γ_2
DAX 30	0.075736	-0.17654	0.027456	0.86582	0.11766
	(2.2719)	(-4.9226)	(2.3775)	(39.5341)	(6.8878)
CAC 40	0.050762	-0.6265	0.008936	0.95577	0.03777
	(1.4146)	(-1.7819)	(1.3806)	(74.9731)	(3.8432)
S&P 500	0.09912	0.064278	0.006227	0.92287	0.074339
	(3.916)	(1.5632)	(2.2765)	(72.1109)	(6.8585)

Table II

AR(1)-GARCH(1,1) *parameters of DAX30, CAC40 and S&P500 return series.*

For these reasons we assume an AR(1)-GARCH(1,1) to model the portfolio returns. However, the residuals of an AR(1)-GARCH(1,1) model generally cannot be considered Gaussian distributed. As a matter of fact, we observe that t_3 and α-stable distributions present the best performance to approximate the empirical distribution of residuals, as confirmed by Kolmogorov-Smirnov and Anderson-Darling distance. From this empirical analysis (see Bertocchi et al. (2004)) it results that α-stable or t_3 distributed innovations would be a reasonable assumption.

4. THE ASSET ALLOCATION PROBLEM

In order to evaluate the impact of different distributional approximations for future returns in the investor's portfolio choice, we consider a dynamic asset allocation approach very similar to that proposed by Boender (1997) and Tokat, Rachev and Schwartz (2003). We generate different initial asset allocations then simulated in the future by using the economic scenarios generated under the Gaussian, the stable and the t_3 assumptions for the innovations of the time series models. We are interested in maximizing the expected utility of the final wealth, W_T, at time horizon *T*.

4.1. *The choice of utility functions*

We assume that investors wish to maximize three different functionals of final wealth which differ for their risk measures and for the risk aversion coefficient *c*.

1. The *mean-VaR utility functional*

$$U_{(1)}(W_T) = E(W_T) - cVaR_{\alpha}(W_T) \tag{10}$$

where $VaR_{\alpha}(W_T) = -\inf\left(z / P\left(W_T \le z\right) > \alpha\right)$ and α represents the maximum probability of loss that the investor would accept.

2. The *mean-CVaR utility functional*

$$U_{(2)}(W_T) = E(W_T) - cCVaR_{\alpha}(W_T) \tag{11}$$

where $CVaR_{\alpha}(W_T) = -E\left(W_T / -W_T \ge VaR_{\alpha}\left(W_T\right)\right)$. In particular we recall that the conditional value at risk is a coherent risk measure.

3. The *mean-dispersion utility functional*

$$U_{(3)}(W_T) = E(W_T) - cE(|W_T - E(W_T)|^q) \tag{12}$$

with power *q*>0 is equivalent to maximizing the utility functional $aE(W_T) - bE\left(|W_T - E(W_T)|^q\right)$, assuming *c*=(*b/a*) for every *a,b*>0. Thus, $E\left(|W_T - E(W_T)|^q\right)$ represents a particular risk measure of portfolio loss which satisfies the main characteristics of the classical dispersion measures. By solving optimally this allocation problem, the investor implicitly maximizes the expected mean of the increment wealth aW as well as minimizes the individual risk $bE\left(|W_T - E(W_T)|^q\right)$. In particular, when *q*=2, the maximization of the utility functional motivates the mean variance approach in terms of preference relations.

4.2. *Solving an optimization model*

We shall consider that the decision makers with utility functions (10), (11), (12) invest their wealth among the three indexes S&P500, DAX30 and CAC40 plus a riskless asset (that we assume to be equal to 6% a.r.). The investor calibrates her/his portfolio at every decision stage using the fixed-mix allocation rule. This strategy is often used by financial practitioners, even if it is an optimal strategy only under some regularity conditions. The fixed-mix strategy maintains a fixed exposure on the different assets by requiring the purchase of stocks as they decrease in value and the sale of stocks as they increase in value.

Therefore, among the initial fixed-mix portfolios the investor will choose the portfolio $\mathbf{x}' = [x_1, ..., x_n]$ which maximizes the utility functional associated $U_{(k)}(W_T^{\mathbf{x}})$, i.e.

$$\max_{\mathbf{x}} U_{(k)}(W_T^{\mathbf{x}}) \quad s.t. \; x_i \ge 0, \; \sum_{i}^{n} x_i = 1 \tag{13}$$

where $U_{(k)}(W_T^{\mathbf{x}})$, is given, in turn, by equations (10), (11), (12),

$$W_{s,T}^{\mathbf{x}} = \sum_{t=1}^{T} R_{s,t}^{\mathbf{x}} \tag{14}$$

is the final wealth for portfolio *x* under the scenario $s \in \{1,\ldots,S\}$ and

$$R_{s,t}^{\mathbf{x}} = \mathbf{x}'\mathbf{R}_{s,t} \tag{15}$$

is the return of portfolio **x** under the scenario *s* at time *t*, $\mathbf{R}'_{s,t} = \left[R_{1,s,t},\ldots,R_{n,s,t}\right]$ where $R_{i,s,t}$ is the rate of return of *i*-th asset under the scenario *s* at time *t*;

$$E(W_T^{\mathbf{x}}) = \frac{1}{S}\sum_{s=1}^{S} W_{s,T}^{\mathbf{x}} \tag{16}$$

is the expected value of the final wealth for portfolio **x**; and

1. $VaR_\alpha(W_T^{\mathbf{x}})$ is the opposite of the $\lceil \alpha S \rceil$-th observation among the *S* ordinate scenarios of final wealth $W_T^{\mathbf{x}}$.
2. $CVaR_\alpha(W_T^{\mathbf{x}}) = \frac{-1}{\lceil \alpha S \rceil}\sum_{s=1}^{\lceil \alpha S \rceil} W_{s,T}^{\mathbf{x}}$ where we assume that the scenarios $W_{s,T}^{\mathbf{x}}$ are ordered in increasing way.
3. $E\left(\left|W_T^{\mathbf{x}} - E\left(W_T^{\mathbf{x}}\right)\right|^q\right) = \frac{1}{S}\sum_{s=1}^{S}\left|W_{s,T}^{\mathbf{x}} - E\left(W_T^{\mathbf{x}}\right)\right|^q$.

These are the three possible risk measures associated to portfolio **x** for the three functionals. We assume that the investor has a temporal horizon of 5 periods, i.e. T=5.

5. COMPUTATIONAL RESULTS AND SCENARIO GENERATION

In order to solve portfolio problem (13), we consider two admissible procedures:

1. using a grid of admissible portfolios,
2. using the simulated annealing procedure in order to obtain the global maximum.

Grid methodology

We consider 21^3 initial asset allocations corresponding to 21 different admissible weights x_i (starting from 0% to 100% with a step of 5%) for the *i*-th asset *i*=1,2,3,4. For each portfolio $R_{p,t} = \mathbf{x}'\mathbf{R}_t$ we estimate the AR(1)-GARCH(1,1) parameters. Then, we simulate the allocations in the future by using the economic scenarios generated by sampling randomly from Gaussian, α-stable and t_3 innovation distributions. For each sample, we simulate 5000 future economic scenarios at daily intervals. Thus, every set of scenarios is generated by assuming that residuals of the variables are i.i.d. normal, i.i.d. stable, i.i.d. t_3 distributed. The horizon of interest is 5 days. Finally, we solve the optimization problem (13). This analysis gives us an approximation of the optimal expected utility considering the three different distribution families. We observe that the volatility of scenarios changes deeply as the distributional assumption for the residuals of portfolios changes. It appears clearly that α-stable scenarios are more volatile than t_3 scenarios, and t_3 scenarios are more volatile than the Gaussian ones.

Simulated annealing methodology

The previous methodology considers only a grid of all admissible portfolios. Observe that the assumption of an AR(1)-GARCH(1,1) for each portfolio and each distributional assumption implies that every expected utility function could present more than a local maximum. Thus, the determination of the optimal portfolio is not an easy issue from the computational point of view and what we consider to be the optimum it may not be the global optimum. In order to overcome this problem, we propose to choose the optimal expected utility among simulated scenarios using the simulated annealing algorithm, first introduced by Kirkpatrick (1983). In this case, the computational complexity increases sensibly and for this reason we suggest this approach only when the number of assets is small. In particular, we try to simplify the computation using the simulated annealing program starting from the point found by the grid analysis. Other methods may be used to find the global optimum, as recently mentioned by Biggs (see Biggs (2004)) or LGO software package (see Pinter (1996)) may be useful.

Parameter"c"		Portfolio Composition			Expected utility	Mean	Risk
	DAX 30	CAC 40	S&P 500	Riskfree			
Gaussian generation							
0.04	59.85%	0.00%	40.15%	0.00%	0.2631	0.4211	3.9503
0.06	60.02%	0.00%	39.98%	0.00%	0.1840	0.4211	3.9517
0.08	58.81%	0.00%	41.17%	0.02%	0.1051	0.4207	3.9449
0.2	0.00%	0.00%	0.00%	100.00%	0.0822	0.0822	0
Stable generation							
0.04	15.69%	34.51%	49.80%	0.00%	0.1770	0.3637	4.6672
0.06	5.89%	40.07%	53.95%	0.09%	0.0918	0.3515	4.3285
0.08	0.89%	2.76%	4.46%	91.89%	0.0803	0.1049	0.3077
0.2	0.00%	0.00%	0.00%	100.00%	0.0822	0.0822	0
t_3 generation							
0.04	0.00%	0.00%	100.00%	0.00%	0.2335	0.4252	4.7932
0.06	0.00%	0.00%	99.60%	0.40%	0.1394	0.4266	4.7868
0.08	0.00%	0.00%	5.00%	95.00%	0.0864	0.0993	0.1616
0.2	0.00%	0.00%	0.00%	100.00%	0.0822	0.0822	0

Table III

Optimal allocation for the optimization problem

$$\max E(W_T)-c\,CVaR_{95\%}(W_T)$$

when Gaussian, Student's t_3 and stable non-Gaussian distributional assumptions are considered.

5.1. *Distributional analysis on some optimal portfolios*

As a consequence of the previous analysis we obtain a different optimal portfolio for each utility function (10), (11), (12) as well as different risk aversion coefficients "c". In Tables III, IV and V we report some of the optimal portfolios obtained under the different distributional hypotheses of the return residuals. These tables show that for each utility functionals the portfolio choice consisting of α-stable scenarios is the most diversified and it always considers the index CAC 40 while the other set of scenarios (t_3 and Gaussian) does not. Besides there are strong differences among the allocation choices based on different approaches. The main reason of this different diversification is probably due to the fact that CAC 40 presents the less fat tails among the three risky indexes. Thus, in order to minimize the risk due to the tail distributions, the "stable investor" assumes a positive position on this index, even if CAC 40 is less appealing in terms of the expected return. Note that the investor with t_3 distributed innovations chooses the less diversified optimal portfolios and he/she invests all the wealth only in the riskless and in the S&P 500 index.

In order to evaluate the skewness and the kurtosis of optimal portfolio choices, we consider the MLE *t*□ and the α-stable parameters (see Bertocchi et al. (2004)). This simple analysis shows that all optimal portfolios obtained with α-stable or Gaussian residuals generally present a large index of stability α (about α=1.91) and large degrees of freedom v. These optimal portfolios are less risky than those obtained assuming t_3 innovations.

To help in distinguishing among competing models, we use two goodness-of-fit measures, i.e the Kolmogorov-Smirnoff statistic and the Anderson-Darling statistic. Therefore we can compare the empirical cumulative distribution F_E of the optimal portfolio residuals with either a simulated Gaussian or a non-Gaussian cumulative distribution F. The Kolmogorov-Smirnoff test does not show big differences among the three distributional, indeed, the Anderson Darling statistic agrees upon in rejecting the hypothesis of normality, while t_3 and α-stable distributed residuals generally cannot be rejected (see Bertocchi et al. (2004) for details).

Parameter"c"	Portfolio	Composition			Expected utility	Mean	Risk
	DAX 30	CAC 40	S&P 500	Riskfree			
Gaussian generation							
0.04	60%	0%	40%	0%	0.3006	0.4211	3.0136
0.06	58.75%	0%	41.25%	0%	0.2405	0.4207	3.0026
0.08	58.44%	0%	41.56%	0%	0.1805	0.4205	3.0001
0.2	0%	0%	0%	100%	0.0822	0.0822	0
Stable generation							
0.04	16.33%	34.54%	49.13%	0.00%	0.242	0.3628	3.0206
0.06	5.93%	40.07%	53.92%	0.08%	0.1896	0.3515	2.6995
0.08	8.70%	41.87%	47.14%	2.29%	0.1371	0.3491	2.6503
0.2	0.00%	0.00%	4.99%	95.01%	0.0763	0.097	0.1035
t_3 generation							
0.04	0.00%	0.00%	100.00%	0.00%	0.2893	0.4252	3.3978
0.06	0.00%	0.00%	100.00%	0.00%	0.2213	0.4252	3.3978
0.08	0.00%	0.00%	99.99%	0.01%	0.1556	0.4286	3.4124
0.2	0.00%	0.00%	0.00%	100.00%	0.0822	0.0822	0

Table IV

Optimal allocation for the optimization problem

$$\max E(W_T)-cVaR_{95\%}(W_T)$$

when Gaussian, Student's t_3 and stable non-Gaussian distributional assumptions are considered.

Parameter"c"	Portfolio	Composition			Expected utility	Mean	Risk
	DAX 30	CAC 40	S&P 500	Riskfree			
Gaussian generation							
0.024	60.00%	0.00%	40.00%	0.00%	0.3609	0.4211	2.5068
0.05	59.85%	0.00%	40.15%	0.00%	0.2958	0.4211	2.5057
0.1	49.96%	0.00%	35.03%	15.01%	0.1738	0.3699	1.9607
0.4	0.00%	0.00%	2.50%	97.50%	0.0851	0.0901	0.0125
Stable generation							
0.024	11.30%	38.64%	50.06%	0.00%	0.3013	0.3659	2.6923
0.05	5.89%	40.08%	54.02%	0.01%	0.2404	0.3521	2.234
0.1	8.23%	14.74%	22.19%	54.84%	0.1309	0.2083	0.774
0.4	0.00%	0.00%	4.97%	95.03%	0.0815	0.097	0.0388
t_3 generation							
0.024	0.00%	0.00%	100.00%	0.00%	0.3540	0.4286	3.1078
0.05	0.00%	0.00%	100.00%	0.00%	0.2712	0.4252	3.0807
0.1	0.00%	0.00%	55.68%	44.32%	0.1452	0.2732	1.2798
0.4	0.00%	0.00%	4.99%	95.01%	0.0855	0.0993	0.0344

Table V

Optimal allocation for the optimization problem

$$\max E(W_T)-cE(|W_T-E(W_T)|^{1.5})$$

when Gaussian, Student's t_3 and stable non-Gaussian distributional assumptions are considered.

6. CONCLUSIONS

This paper proposes and compares alternative distributional assumptions for portfolio selection models. First, we analyze the historical series considered in the empirical portfolio selection proposed. The observed skewness and kurtosis together with other empirical tests point out that the returns are generally far to be Gaussian distributed. Moreover, we observe that there is a strong evidence that the indexes returns follow an AR(1)-GARCH(1,1) model. In particular, we consider different distributional assumptions on return innovations because we observe that also residuals of the AR(1)-GARCH(1,1) model generally cannot be considered as Gaussian distributed.

Secondly, we propose a comparison made among the α-stable, Gaussian and the t_3 distributed portfolio residuals assuming three different allocation problems. We find that the tail behavior of different distributional approaches could imply substantial differences in the asset allocation. This analysis indicates that the α-stable allocation is more risk preserving than the other ones because it presents the greatest diversification in the portfolio composition and it considers the component of risk due to the fat tails. On the other hand, all the statistical tests have pointed out that the α-stable and the t_3 present the best approximation to the data. Taken into account that the α stable and t_3 approaches are more adherent to the reality of the market, we have obtained models that improve the performance measurements with the distributional assumption: the α stable approach which is more risk preserving and the t_3 approach which is riskier but takes into more account the final expected wealth.

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