Karlsruhe Institute for Technology

Chair of Economics, Statistics and Mathematical Finance

School of Economics and Business Engineering

Prof. Dr. W.-D. Heller

The Normal Tempered Stable Distribution for synthetic CDO pricing

Working Paper

Philipp Ehrler

Dr. Markus Höchstötter

Contents

1. Introduction

Returns of financial assets exhibit skewness and leptokurtosis revealing risk aversion and the existence of extreme events. To take these circumstances into account, the family of α-stable distributions has been heavily discussed and applied throughout the past years (e.g. Rachev and Mittnik (2000)). In addition to location and shift parameters α-stable distributions provide the ability to adjust for skewness and kurtosis. But with these advantages new challenges come along. With the exception of the gaussian distribution, one crucial drawback of the α-stable distributions in many applications is their generally infinite variance. Moreover, parameterizations that feature high excess kurtosis (i.e. small α) often come with infinite expectation. Both is due to the fact that an α-stable distribution's density function decays only polynomially.

As introduced by Menn and Rachev (2009), and applied in Schmitz et al. (2010) smoothly truncated α-stable (STS) distributions can be used to cope with the non-existence of moments of order $p > \alpha$. STS distributions are obtained by truncating the heavy tails of an α-stable density function and replacing them by the thin tails of two appropriately chosen gaussian distributions.

A more recent approach is to temper the Lévy measure ν of a symmetric α-stable distribution, i.e. by multiplying it with a suitable function that decreases in the absolute value of the argument of the Lévy measure. Thereby, large jumps occur less frequently than under the stable assumption (we will discuss this in more detail below). Following the words of Rosinski (2007), the properties of the resulting class of tempered stable distributions, respectively a tempered stable process can be described as follows: "In a short time frame it is close to an α-stable process, while in a long time frame it approximates a Brownian motion". Based on the work of Schmitz et al. (2010) we will apply heavy tailed distributions to the pricing of synthetic CDOs and assess the impact of a tighter dependence structure. A calibration of iTraxx Europe Series 7 on-the-run CDO tranche spreads will be done in the second part of this work to challenge the models.

2. Normal Tempered Stable Process

2.1. Univariate Definition

There are many ways to introduce the Normal Tempered Stable (NTS) distribution: Rachev et al. (2011) provide the tempering function, Kim et al. (2008) apply exponential tilting to the symmetric Modified Tempered Stable distribution, whereas we will stick to Rachev et al. (2011) and Barndorff-Nielsen and Levendorskii (2001) and present the NTS distribution as the building block of a time-changed Brownian motion. The underlying idea behind subordination is as follows: Instead of evaluating a stochastic process Y_t^1 at the real time t (i.e. t is non-stochastic) we will introduce a random variable T_t and define a new process $Y_t^2 = Y_{T_t}^1$. This concept is referred to as subordination and the random time T_t is called subordinator of the process (in many applications T_t is called *trading or business time*).

2.1.1. Classical Tempered Stable subordinator

Following the notation used in Rachev et al. (2011), the support of the α-stable distribution with $\beta = 1$, $\mu = 0$ and $0 < \alpha < 1$ is the positive real line and hence the corresponding α-stable process qualifies as a subordinator (Nolan (2012)). If we multiply the Lévy measure of the α-stable subordinator by the tempering function of the CTS we end up with the following expression:

$$\nu(dx) = \frac{Ce^{-\theta x}}{x^{\alpha/2+1}} 1_{x>0}.$$

Setting $\gamma = \int_0^1 x\nu(dx)$ we ensure that the process has only finite variation and a positive drift. Moreover, if we set $C = \frac{1}{\Gamma(1-\frac{\alpha}{2})\theta^{\frac{\alpha}{2}-1}}$ then

$$E[T_t] = g'(0) = tC\Gamma(1 - \frac{\alpha}{2})\theta^{\frac{\alpha}{2}-1} = t$$

holds, i.e. under the expectation operator T_t will equal real time. We will call the resulting process the Classical Tempered Stable (CTS) subordinator with parameters (α, θ). Its characteristic function is given by

$$\phi_{T_t}(u) = \exp((-\frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha}((\theta - iu)^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}}))t).$$

2.1.2. Normal Tempered Stable distribution

Let μ, $\beta \in \mathbb{R}$, $\gamma > 0$, define B_t as a Brownian Motion and T_t as a CTS subordinator with parameters $(\alpha,\ \theta)$. Then, we can define the following stochastic process as the Normal Tempered Stable Process:

$$X_t = \mu t + \beta(T_t - t) + \gamma B_{T_t}. \tag{2.1}$$

The characteristic function of X_t is given by:

$$\phi_{X_t}(u) = e^{iu(\mu-\beta)t}\phi_{T_t}(\Psi_{N(\beta,\gamma^2)}) = \exp(iu(\mu-\beta)t - \frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha}((\theta - i\beta u + \frac{\gamma^2 u^2}{2})^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}})t). \tag{2.2}$$

The random variable X_t has an expected value $E[X_t] = \mu t$ and variance $var[X_t] = \gamma^2 t + \beta^2(\frac{2-\alpha}{2\theta})t$. The latter is simply the total variance of all summands in equation 2.1 and thus, T_t and B_{T_t} are uncorrelated. Although there is no linear relationship between T_t and B_{T_t}, still, there exist higher order dependencies. Apparently, the realisation of T_t determines the variance of B_{T_t}. Now, for the practical application in the remainder a standardisation - meaning that $E[X_t] = 0$ and $var[X_t] = t$ hold - would be useful. From the above we deduce that this will only be the case if $\mu = 0$ and $\gamma = \sqrt{1 - \beta^2(\frac{2-\alpha}{2\theta})}$. To avoid any negative number below the radical sign $|\beta| < \sqrt{(\frac{2\theta}{2-\alpha})}$ needs to be satisfied. After all we arrive at the desired definition of a univariate NTS random variable:

Definition 2.1.1. *X_1 follows a Normal Tempered Stable distribution iif X_t is a Normal Tempered Stable process with parameters $(\alpha,\ \theta,\ \mu,\ \beta,\ \gamma)$, where $\alpha \in (0, 2)$, $\theta,\ \gamma > 0$ and $\beta,\ \mu \in \mathbb{R}$.*

2.2. Equivalent representation and convolution

So far, we have introduced the NTS distribution following the Barndorff-Nielsen representation. A second equivalent representation can be found in Rachev et al. (2011):

$$\begin{aligned}
&\phi_{NTS}(u; \tilde{\alpha}, C, \lambda, b, m) \\
&= \exp(ium - iu2^{-\frac{\tilde{\alpha}-1}{2}}\sqrt{\pi}C\Gamma(1 - \frac{\tilde{\alpha}}{2})b(\lambda^2 - b^2)^{\frac{\tilde{\alpha}}{2}-1}\dots \\
&\dots + 2^{-\frac{\tilde{\alpha}+1}{2}}C\sqrt{\pi}\Gamma(-\frac{\tilde{\alpha}}{2})((\lambda^2 - (b + iu)^2)^{\frac{\tilde{\alpha}}{2}} - (\lambda^2 - b^2)^{\frac{\tilde{\alpha}}{2}})),
\end{aligned} \tag{2.3}$$

with $\tilde{\alpha} \in (0, 2)$, $C,\ \lambda > 0$, $|b| < \lambda$, and $m \in \mathbb{R}$. Both representations are equal if the following equalities hold:

$$\tilde{\alpha} = \alpha \tag{2.4}$$

$$\lambda = \sqrt{\frac{2\theta}{\gamma^2} + b^2} \tag{2.5}$$

$$b = \frac{\beta}{\gamma^2} \tag{2.6}$$

$$C = \frac{\sqrt{2}\gamma^\alpha}{\sqrt{\pi}\Gamma(1-\frac{\alpha}{2})\theta^{\frac{\alpha}{2}-1}} \tag{2.7}$$

$$m = \mu \tag{2.8}$$

We obtain the standardized NTS distribution if we set $m = 0$ and

$$C = 2^{\frac{\tilde{\alpha}+1}{2}}(\sqrt{\pi}\Gamma(-\frac{\tilde{\alpha}}{2})\tilde{\alpha}(\lambda^2 - b^2)^{\frac{\tilde{\alpha}}{2}-2}(\tilde{\alpha}b^2 - \lambda^2 - b^2))^{-1}.$$

We conclude this subsection by outlining the convolution of independent NTS random variables as well as the linear transformation of NTS distributed random variables. Under the representation introduced by Rachev et al. (2011) the following rules apply: Given two independent random variables $X \sim NTS(\tilde{\alpha}, \lambda, b, C_1, m_1)$ and $Y \sim NTS(\tilde{\alpha}, \lambda, b, C_2, m_2)$, then their sum will follow the NTS distribution with the following parameters:

$$X + Y \sim NTS(\tilde{\alpha}, \lambda, b, C_1 + C_2, m_1 + m_2). \tag{2.9}$$

Further if $k_1, k_2 \in \mathbb{R}$

$$k_1 + k_2 X \sim NTS(\tilde{\alpha}, \frac{\lambda}{b}, \frac{b}{k_2}, k_2^\alpha C, k_2 m + k_1) \tag{2.10}$$

holds. Under the Barndorff-Nielsen representation the convolution of two independent NTS random variables is a little less convenient. Given $X \sim NTS(\alpha, \beta_1, \gamma_1, \theta_1, \mu_1)$ and $Y \sim NTS(\alpha, \beta_2, \gamma_2, \theta_2, \mu_2)$, $X + Y$ will only follow the NTS distribution if

$$\frac{\gamma_2^2}{\gamma_1^2} = \frac{\theta_2}{\theta_1} = \frac{\beta_2}{\beta_1}.$$

In this case

$$X + Y \sim NTS(\alpha, \beta_1 + \beta_2, \theta_1 + \theta_2, \gamma_{1+2}, \mu_1 + \mu_2), \tag{2.11}$$

where

$$\gamma_{1+2} = (\theta_1 + \theta_2)^{\frac{1}{2}-\frac{1}{\alpha}}(\frac{\gamma_1^\alpha}{\theta_1^{\frac{\alpha}{2}-1}} + \frac{\gamma_2^\alpha}{\theta_2^{\frac{\alpha}{2}-1}})^{\frac{1}{\alpha}}.$$

Finally, if $k_1, k_2 \in \mathbb{R}$

$$k_1 + k_2 X \sim NTS(\alpha, k_2\beta, \theta, k_2\gamma, k_2\mu)$$

holds.

2.3. Semi-closed pdf and multivariate definition

Let X be a NTS distributed random variable. Then we know from above that the following equality holds:

$$X = \mu + \beta(T-1) + \gamma B_T = \mu + \beta(T-1) + \gamma\sqrt{T}\epsilon,$$

with $\epsilon \sim N(0,1)$ and $E[T] = t = 1$. The cumulative density function of X is then given by:

$$F_X(z) = P(X < z) = \int_0^\infty \int_{-\infty}^z \frac{1}{\sqrt{2\pi\gamma^2 s}} e^{-\frac{(x-\mu-\beta(s-1))^2}{2\gamma^2 s}} dx f_T(s) ds, \tag{2.12}$$

where f_T denotes the density function of the random variable T. The probability density function f_T can be efficiently obtained by the use of the inverse Fourier transform (see Scherer et al. (2010))[1] :

$$f_T(t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-iut}\phi_T(u)du, \tag{2.14}$$

For the multivariate case we distinguish between two different configurations. We will first discuss the case where several random variables are driven by the same subordinator and subsequently when each random variable is time-changed by its own subordinator. Assume first that the random variables are driven by the same subordinator:

$$X_i = \mu_i + \beta_i(T-1) + \gamma_i\sqrt{T}\epsilon_i,$$

with $i \in \{1, 2, \ldots, d\}$ and $Corr(\epsilon_i, \epsilon_j) = \rho_{i,j}$. Then the linear correlation coefficient between X_i and X_j is given by

$$Cov[X_i, X_j] = \beta_i\beta_j\frac{(2-\alpha)}{2\theta} + \gamma_i\gamma_j\rho_{i,j}.$$

We note that conditional on T, the covariance between X_i and X_j would be $\gamma_i\gamma_j\rho_{i,j}$. However, as T affects the mean and variance of the conditional (Gaussian) random variable the covariance increases by $\beta_i\beta_j\frac{(2-\alpha)}{2\theta}$ in the unconditional case. Hence, we can calculate the cumulative distribution function for the multivariate NTS distribution as

$$F(a_1, \ldots, a_d) = P(X_1 < a_1, \ldots, X_d < a_d) = \int_0^\infty G(t) f_{T_1} dt, \tag{2.15}$$

where

$$G(t) = \int_{-\infty}^{\frac{a_1-\mu_1-\beta_1(t-1)}{\gamma_1\sqrt{t}}} \cdots \int_{-\infty}^{\frac{a_d-\mu_d-\beta_d(t-1)}{\gamma_d\sqrt{t}}} f_\epsilon(x_1, \ldots, x_d) dx_1 \ldots dx_d. \tag{2.16}$$

[1] Please note that one could directly determine F_X by the following equality given in Rachev et al. (2011):

$$F_X(x) = \frac{e^{x\rho}}{\pi}\Re\left(\int_0^\infty e^{-ixu}\frac{\phi_X(u+i\rho)}{\rho - iu}du\right). \tag{2.13}$$

This will only hold if X is an infinitely divisible random variable and if there is a $\rho > 0$ such that $|\phi_X(u + i\rho)| < \infty$ for all $u \in R$ (see Rachev et al. (2011), p.138). However, this becomes less attractive in the multivariate case where the FFT grid grows polynomial but still needs to exhibit a fine structure to maintain accuracy.

And the probability density function can written as

$$f_X(x) = \int_0^\infty \frac{1}{(2\pi)^{\frac{d}{2}} |\Sigma(t)|^{\frac{1}{2}}} \exp(-\frac{1}{2}(x - m(t))'(\Sigma(t))^{-1}(x - m(t)))f_{T_1}dt, \tag{2.17}$$

where $x = (x_1, \ldots, x_d)$, $m(t) = \mu+\beta(t-1) = \{\mu_1+\beta_1(t-1), \ldots, \mu_d+\beta_d(t-1)\}$ and $\Sigma(t) = [\gamma_k\gamma l\rho_{k,l}t]_{k,l\in\{1,\ldots,d\}}$. If we now consider the case where each random variable is driven by its own subordinator and assume that all subordinators are mutually independent, e.g.

$$X_i = \mu_i + \beta_i(T_i - 1) + \gamma_i\sqrt{T_i}\epsilon_i,$$

with $i \in \{1, 2, \ldots, d\}$ and $Corr(\epsilon_i, \epsilon_j) = \rho_{i,j}$, then the covariance between X_i and X_j is:

$$Cov[X_i, X_j] = \gamma_i\gamma_j E[\sqrt{T_i}]E[\sqrt{T_j}]\rho_{i,j}.$$

The multivariate cumulative density function can then be written as:

$$F(a_1, \ldots, a_d) = P(X_1 < a_1, \ldots, X_d < a_d) \tag{2.18}$$
$$= \int_0^\infty \cdots \int_0^\infty \Phi_2(\frac{a_1 - \mu_1 - \beta_1(t_1 - 1)}{\gamma_1\sqrt{t_1}}, \ldots, \frac{a_d - \mu_d - \beta_d(t_d - 1)}{\gamma_d\sqrt{t_d}}, \Sigma)f_{T_1} \ldots f_{T_d}dt_1 \ldots dt_d,$$

where $\Sigma = [\rho_{i,j}]_{i,j\in\{1,\ldots,d\}}$. At the end of this paper we have plotted the contour of the pdf of a stdNTS distributed random variable with a common subordinator (figure 4.1.B)) and mutually independent subordinators (Figure 4.1.A)). Figure 4.2 shows the difference between the two pdfs (A - B). Clearly, in the case of a common subordinator more probability mass is located in areas where both realizations of the marginal random variables exhibit similar values in absolute terms. This is hardly surprising as both marginals are simulaneously time-changed by the same subordinator, i.e. their variance is always identical. Besides, figure 4.3 reveals that the tendency for joint extreme events remains high in both cases. For reasons of illustration we have added the *lower tail dependence coefficient* λ_L for the bivarate gaussian distribution. In contrast to the gaussian copula the lower tail dependence coefficient[2] of the NTS copula remains significant in both cases.

[2]
$$\lambda_L = \lim_{u\to 0} P(X \leq F_X^{-1}(u)|Y \leq F_Y^{-1}(u)) \tag{2.19}$$

3. Synthetic Credit Default Obligations

Assume a portfolio of n equally weighted CDS contracts. Then the percentage loss at time t is given by

$$L_t = \frac{1-R}{n} N_t, \tag{3.1}$$

where R describes the deterministic recovery rate that is assumed to be the same for all single names and

$$N_t = \sum_{i=1}^{n} 1_{\{\tau_i \leq t\}} \tag{3.2}$$

counts the number of defaults that have occured until t. For simplicity reasons we assume that compensation payments are only settled on one of the premium payment days $T_1, \dots, T_m$ and that the portfolio loss has been "tranched" into several disjoint tranches $[a, b]$, where we will call a attachment and b detachment point of the tranche. In order to derive a pricing equation, we define the tranche loss as

$$L_t^{[a,b]} = (L_t - a)^+ - (L_t - b)^+$$

and summarize the present value of all contingent payments to be done by the protection seller (so-called default leg)

$$\text{default leg} := \sum_{T_m > t} \frac{B(t, T_m)}{b-a} E^{\mathbb{Q}}[L_{T_m}^{[a,b]} - L_{T_{m-1}}^{[a,b]} | \mathcal{F}_t], \tag{3.3}$$

where $E^{\mathbb{Q}}$ defines the expectation operator under the risk-neutral probability measure $\mathbb{Q}$ and $B(t, T_m)$ is the price of a zero-coupon bond with maturity date T_m at time t. Analogously, the present value of all expected premium payments is defined as the *premium leg*

$$\text{premium leg} := s_0^{[a,b]} \sum_{T_m > t} \frac{B(t, T_m)}{b-a} (T_m - T_{m-1}) E^{\mathbb{Q}}[1 - L_{T_m}^{[a,b]} | \mathcal{F}_t]. \tag{3.4}$$

To obtain the fair market spread $s_t^{[a,b]}$ that balances the two legs we set both legs equal and solve for $s_t^{[a,b]}$

$$s_t^{[a,b]} := \frac{\sum_{T_m>t} B(t,T_m)E^{\mathbb{Q}}[(L_{T_m}^{[a,b]} - L_{T_{m-1}}^{[a,b]})|\mathcal{F}_t]}{\sum_{T_m>t} B(t,T_m)E^{\mathbb{Q}}[(T_m - T_{m-1})(1 - L_{T_m}^{[a,b]})|\mathcal{F}_t]}. \tag{3.5}$$

3.1. Gaussian Copula Model

The Gaussian latent variable model published in Vasicek (1987) has been the underlying concept used in most pricing and risk management models. Likewise, it will be the foundation for all models discussed in this paper.

In the Gaussian latent variable model the time of default is modelled as the first arrival time of a homogenous poisson process with constant intensity λ_D. Interarrival times, are therefore, exponentially distributed and the surival probability of credit i with time horizon T, given that no default has occured before t, is

$$Q_i(T) = \exp^{\lambda_D(T-t)}. \tag{3.6}$$

As a next step we introduce a standard gaussian random variable A_i and assume that a credit event occurs before time T if A_i is below a threshold $C_i(T)$. If we use CDS spreads to determine implicit default probabilities [1], we can write the probability of default before time T in mathematical terms as

$$P(\tau_i \leq T) = P(A_i \leq C_i(T)) = 1 - Q_i(T), \tag{3.7}$$

where τ_i describes the time of default of credit i. Eventually, with $A_i \sim N(0,1)$

$$C_i(T) = \Phi^{-1}(1 - Q_1(T)), \tag{3.8}$$

follows.

Now to create a dependence structure between various credits in a portfolio a one-factor modell is used

$$A_i = v_i Z + \sqrt{1 - v_i^2}\epsilon_i, \tag{3.9}$$

with both Z and ϵ_i being independent standard gaussian random variables. Consequently the risk of credit i can be split into two parts. Z, defined as the market factor influencing all credits, and ϵ_i representing idiosyncratic risk specific to each credit i. Depending on $v_i \in [0,1]$ both factors are weighted. At the same time, the weighting in the one-factor structure ensures that $A_i \sim N(0,1)$. Due to the fact that all random variables are gaussian the dependance structure between two credits i and j can be properly described by the linear correlation coefficient $\rho = v_i v_j$.

[1] $\lambda_D = \frac{CDS_{spread}}{1-R}$, see Hull (2002) page 484

3.1.1. Gaussian Large Homogeneous Pool Model

Assuming a homogeneous credit portfolio, the limiting case of the Vasicek model where the number of single names n goes to infinity is called the LHP model. Though simplifications are remarkable, the LHP model first introduced in Li (2000) has become the standard market model [2]. In a homogeneous portfolio (i.e. $v = v_i, \forall i$) the probability of default before time T conditional on the market factor Z is

$$p(T|Z) = \Phi(\frac{C(T) - vZ}{\sqrt{1-v^2}}), \tag{3.10}$$

which is also the case for all other names. Conditional on Z all credits in the portfolio are independent, and the conditional loss distribution for the portfolio is the binomial distribution. Apparently the variance falls as $\mathcal{O}(n^{-1})$, and, consequently, the conditional loss distribution tends to unit point mass of probability located at the expected portfolio loss. By integrating over Z we can calculate the unconditional portfolio loss distribution: In the limiting case the portfolio loss conditional on Z is given by

$$L(T|Z) = (1-R)p(T|Z) = (1-R)\Phi(\frac{C(T) - vZ}{\sqrt{1-v^2}}). \tag{3.11}$$

The portfolio loss distribution can then be written as

$$F(K|Z) = P(L(T|Z) \leq K) = P((1-R)\Phi(\frac{C(T) - vZ}{\sqrt{1-v^2}}) \leq K). \tag{3.12}$$

Isolating the source of uncertainty in the argument

$$Z \geq A(K) \tag{3.13}$$

where

$$A(K) = \frac{1}{v}(C(T) - \sqrt{1-v^2}\Phi^{-1}(\frac{K}{1-R})),$$

and recalling that $Z \sim N(0,1)$, the cumulative distribution function is defined as

$$F(K) = 1 - \Phi(A(K)). \tag{3.14}$$

To price synthetic CDO tranches as in equation 3.5, we need to derive the expected tranche loss from the results above. Indeed, the tranche survival probability is a non-linear function of the portfolio loss

$$Q(t, K_1, K_2) = 1 - \frac{E[\min(L(t), K_2) - \min(L(t), K_1)]}{(K_2 - K_1)}, \tag{3.15}$$

where K_1 and K_2 describe the attachment and detachment point of the tranche respectively. We can separate the summands in the argument of the expectation operator and

[2] We note that a wide range of extensions exist.

cut down the problem to

$$E[\min(L(T),K)] = E[L(T)1_{L(T)<K}] + KE[1_{L(T)\geq K}],$$

where we know that

$$E[1_{L(t)\geq K}] = P(L(t)\geq K) = \Phi(A(K)).$$

The first summand appears to be a little more tricky. However, we can show that

$$E[L(T)1_{L(t)<K}] = (1-R)\int_{A(K)}^{\infty}\Phi(\frac{C(T)-vZ}{\sqrt{1-v^2}})\phi(Z)dZ = (1-R)\Phi_2(C(t),-A(K),-v).$$

3.2. Extensions using NTS

Earlier on we mentioned that the *lower tail dependence* of the gaussian copula is asymptotically independent. To overcome this drawback we will now present two extensions with stdNTS distributed marginals.

3.2.1. NTS Copula Model I

Our first extension simply adds the idea of stochastic time to the LHP model. As a result, Z and ϵ_i follow a stdNTS distribution. Moreover, the random variable Z can be skewed, and the dependence structure between the single names can take on new forms.

We start by defining the same one-factor model

$$A_i = vZ + \sqrt{1-v^2}\epsilon_i, \tag{3.16}$$

with the difference that $Z \sim stdNTS(\alpha,\theta,\beta)$ and $\epsilon_i \sim stdNTS(\alpha,\theta,0)$. We assume Z and ϵ_i to be time-changed by the same subordinator $T \sim subCTS(\alpha,\theta)$, and breakdown the random variables into their components

$$Z = \beta(T-1) + \sqrt{1-\beta^2(\frac{2-\alpha}{2\theta})}\sqrt{T}\xi_Z \tag{3.17}$$

$$\epsilon_i = \sqrt{T}\xi_i \tag{3.18}$$

As defined in equation 2.1, ξ_z and ξ_i are two independent standard Gaussian random variables.

Although the linear correlation coefficient $\rho_{\xi_Z,\xi_i} = 0$, Z and ϵ_i are not independent as they are both driven by the same subordinator. Using the results of chapter 2.2 we can easily show that A_i still follows a stdNTS distribution with the following parameterization

$$A_i \sim stdNTS(\alpha,\theta,\tilde{\beta}), \tag{3.19}$$

where $\tilde{\beta} = v\beta$. The linear correlation coefficient between two credits i and j is again given by $\rho_{i,j} = v^2$.

Following the procedure used for the Gaussian LHP model we obtain the portfolio loss distribution in the limiting case as follows:

$$\begin{aligned} P(\tau_i \leq T) =& P(A \leq C(T)) \\ =& P(vZ + \sqrt{1-v^2}\epsilon \leq C(T)) \\ =& P(v\beta(T-1) + v\gamma\sqrt{T}\xi_z + \sqrt{1-v^2}\sqrt{T}\xi_i \leq C(T)) \\ =& \Phi_{N(0,1)}(\frac{C(T) - v\beta(T-1) - v\gamma\sqrt{T}\xi_Z}{\sqrt{1-v^2}\sqrt{T}}). \end{aligned} \tag{3.20}$$

As a result, for a fixed Z (i.e. ξ_z) and T, the portfolio loss is analogous to equation (3.11)

$$(1-R)\Phi_{N(0,1)}(\frac{C(T) - v\beta(T-1) - v\gamma\sqrt{T}\xi_Z}{\sqrt{1-v^2}\sqrt{T}}). \tag{3.21}$$

Then the portfolio loss distribution conditional on T is

$$\begin{aligned} F(K|T) =& P((1-R)\Phi_{N(0,1)}(\frac{C(T) - v\beta(T-1) - v\gamma\sqrt{T}\xi_Z}{\sqrt{1-v^2}\sqrt{T}}) \leq K) \\ =& P(\xi_z > \frac{\Phi^{-1}(\frac{K}{1-R}\sqrt{1-v^2}\sqrt{T} - C(T) + v\beta(T-1)}{-v\gamma\sqrt{T}}) \\ =& 1 - \Phi_{N(0,1)}(\frac{\Phi^{-1}(\frac{K}{1-R}\sqrt{1-v^2}\sqrt{T} - C(T) + v\beta(T-1)}{-v\gamma\sqrt{T}}) \\ =& 1 - \Phi_{N(0,1)}(A(K,T)). \end{aligned} \tag{3.22}$$

Taking the randomness of T into account

$$F(K) = \int_0^\infty (1 - \Phi_{N(0,1)}(A(K,T))f(T)dT. \tag{3.23}$$

3.2.2. NTS Copula Model II

Loosely speaking, Z represents systematic risk and ϵ_i idiosyncratic risk. However, in the above proposed extension Z and ϵ_i depend on the same subordinator. Therefore, if T is large both systematic and idiosyncratic risk are higher, et vice versa. To overcome this drawback we will now use two independent subordinators for both random variables. For reasons of clarity we will use the NTS representation introduced by Rachev et al. (2011).

Again, let the standardized firm value be described by a one-factor model

$$A_i = vZ + \sqrt{1-v^2}\epsilon_i, \tag{3.24}$$

where $Z \sim stdNTS(\alpha, \lambda v, bv, c_1)$ and $\epsilon_i \sim stdNTS(\alpha, \lambda\sqrt{1-v^2}, b\sqrt{1-v^2}, c_2)$. Z and ϵ_i are independent, and we can determine the distribution of A_i using the results of chapter 2.2:

$$A_i \sim stdNTS(\alpha, \lambda, b, c_1 v^\alpha + c_2(1-v^2)^{\frac{\alpha}{2}}.$$

Further, $\rho_{A_i,A_j} = v^2$, even though linear correlation will not be sufficient to describe the dependence structure between both. Conditional on Z the expected portfolio loss can be written as

$$E[L(T)|Z] = (1-R)p(T|Z) = (1-R)\Phi_{\epsilon_i}(\frac{C(T)-vZ}{\sqrt{1-v^2}}). \tag{3.25}$$

As $n \to \infty$ the variance of $L(T|Z)$ goes to 0 and again the conditional loss distribution tends to unit point mass of probability located at the expected portfolio loss. We obtain the unconditional portfolio loss distribution following the same procedure as used earlier on

$$\begin{aligned} F(K) &= P(L(T \leq K) \\ &= P((1-R)\Phi_{\epsilon_i}(\frac{C(T)-vZ}{\sqrt{1-v^2}}) \leq K) \\ &= P(Z \geq \frac{C(T) - \Phi_{\epsilon_i}^{-1}(\frac{K}{1-R})\sqrt{1-v^2}}{v}) \\ &= 1 - \Phi_Z(\frac{C(T) - \Phi_{\epsilon_i}^{-1}(\frac{K}{1-R})\sqrt{1-v^2}}{v}). \end{aligned} \tag{3.26}$$

3.3. Discussion

In this section we will sketch out the main differences between the three models. In a first step, we will outline how to obtain the joint default probabilities, and, subsequently, use these results to cast a first glance on the abilities and characteristics of all three models. Our analysis will be mainly based on the bivariate case. Yet, the results can be easily extended for a higher dimensional setting. In particular, for $n \to \infty$ we obtain the respective LHP model. To visualize the results under the LHP assumption we complete this chapter by exploring the respective portfolio loss density functions.

We begin with the gaussian model where the joint default probability of two credits with time horizon T is given by

$$\begin{aligned} &P(\tau_1 \leq T, \tau_2 \leq T) \\ =&P(A_1 \leq C_1(T), A_2 \leq C_2(T)) \\ =&P(vZ + \sqrt{1-v^2}\epsilon_1 \leq C_1(T), vZ + \sqrt{1-v^2}\epsilon_2 \leq C_2(T)) \\ =&\int_{-\infty}^{\infty} \Phi(\frac{C_1(T)-vZ}{\sqrt{1-v^2}})\Phi(\frac{C_2(T)-vZ}{\sqrt{1-v^2}})\phi(Z)dZ \\ =&\Phi_2(C_1(T), C_2(T), \rho = v^2). \end{aligned} \tag{3.27}$$

For the NTS case with a common subordinator (NTS 1) we recall that the covariance between two credits is given by

$$cov(A_1, A_2) = \rho = v^2 \underbrace{(\beta^2(\frac{2-\alpha}{2\theta}) + \gamma^2)}_{=1},$$

where the first summand stems from the fact that both random variables are subordinated by the same stochastic process, and the second summand is due to the common market factor Z. With this in mind, we can define the joint default probability as

$$\begin{aligned}
&P(\tau_1 \le T, \tau_2 \le T)\\
=&P(A_1 \le C_1(T), A_2 \le C_2(T))\\
=&P(vZ + \sqrt{1-v^2}\epsilon_1 \le C_1(T), vZ + \sqrt{1-v^2}\epsilon_2 \le C_2(T))\\
=&\int_0^\infty \int_{-\infty}^\infty \Phi(\frac{C_1(T) - v\beta(T-1) - v\gamma\sqrt{T}z}{\sqrt{1-v^2}\sqrt{T}})\Phi(\frac{C_2(T) - v\beta(T-1) - v\gamma\sqrt{T}z}{\sqrt{1-v^2}\sqrt{T}})\ldots \\
&\ldots \phi(z)dzf(t)dt\\
=&\int_0^\infty \int_{-\infty}^{\frac{C_1(T)-\tilde{\beta}(t-1)}{\tilde{\gamma}\sqrt{T}}} \int_{-\infty}^{\frac{C_1(T)-\tilde{\beta}(t-1)}{\tilde{\gamma}\sqrt{T}}} \phi_2(C_1(T), C_2(T), v^2\gamma^2)dx_1dx_2f(t)dt\\
=&\Phi_2^{NTS}(C_1(T), C_2(T), v^2\gamma^2)_{(\alpha,\theta,\tilde{\beta},\tilde{\gamma})},
\end{aligned} \tag{3.28}$$

where $\tilde{\beta} = v\beta$. Please note that as all variables are standardized we have $\gamma = \sqrt{1-\beta^2(\frac{2-\alpha}{2\theta})}$ and $\tilde{\gamma} = \sqrt{1-\tilde{\beta}^2(\frac{2-\alpha}{2\theta})}$.

Finally, we obtain the following for the NTS model with mutually independent subordinators (NTS 2):

$$\begin{aligned}
&P(\tau_1 \le T, \tau_2 \le T)\\
=&P(A_1 \le C_1(T), A_2 \le C_2(T))\\
=&P(vZ + \sqrt{1-v^2}\epsilon_1 \le C_1(T), vZ + \sqrt{1-v^2}\epsilon_2 \le C_2(T))\\
=&\int_{-\infty}^\infty \Phi_{\epsilon_1}^{NTS}(\frac{C_1(T) - vz}{\sqrt{1-v^2}})\Phi_{\epsilon_2}^{NTS}(\frac{C_2(T) - vz}{\sqrt{1-v^2}})\phi_Z^{NTS}(z)dz,
\end{aligned} \tag{3.29}$$

where we recommend to evaluate the integral nummerically by first calculating the cumulative distribution function Φ_ϵ^{NTS} and the probability density function ϕ_Z^{NTS} by using the good properties of the fast fourier transformation.

Having learned about the mathematical foundations we can now move on to compare the three models. In appendix B we illustrate the differences between the stdNTS and the standard gaussian distribution. The same applies for the bivariate joint default probability distribution in the NTS 1 model. In contrast to the bivariate standard gaussian distribution the bivariate standard NTS distribution provides more mass in the center and in the tails.

Comparing equation 2.27 and 2.28 reveals why this is true. For $\beta = 0$ (i.e. $\gamma = 1$) equation 2.27 is equivalent to 2.28 except that it additionally features stochastic variance. The consquences are as follows: If T is large, the variance of all random variables increases, i.e. high realizations in absolute terms become more likely. For smaller values of T the opposite is the case. All in all, this leads to the above mentioned shifts of probability mass away from the legs (see figure 4.4). Of course, for values of β not equal to zero the NTS 1 model exhibits a skewed density function. Comparing the gaussian model with the NTS 2 model the effects are similar - at least in qualitative terms.

It is hardly surprising that both NTS models exceed the gaussian model in terms of tight dependence and tail probability. Needless to say that both feature a higher *lower tail dependence coefficient* λ_L than the gaussian model. However, as ρ increases the NTS 2 model excessively outperforms the NTS 1 model in the tail (see figure 4.5). This becomes more evident as we plot the contour of the copula functions for both models (see figure 4.6). Apparently, the NTS 2 model exhibits a tighter dependence structure than the NTS 1 model. The reason for this is simple. As outlined before, using a common subordinator for all variables destroys the idea of two independent random variables where one represents systematic and the other idiosyncratic risk. As a result, both random variables will simultaneously possess a high variance if T is large, and a low variance when T is small. Now - for instance - assuming a large realization of T and Z when ρ is not excessively high so that the impact of the ϵ_i's are still significant, then on average (given $b = 0$) half of the ϵ's would reduce the large realization of Z, while the other half would amplify it. This is worth noting as the variance of each ϵ_i is high (i.e. large T). Hence, the contour plot of the copula function peaks less in the tails as we would not expect to have such a high probability for an adverse effect in the NTS 2 model. Knowing that the default threshold $C(T)$ will generally be negative for a reasonable default rate, i.e. small or medium sized negative realizations of A_i will not result in a default, makes the impact of the common subordinator even more influential.

We can visualize the associated effects on the pricing of synthetic CDOs by looking at the portfolio loss density function under the LHP assumption. Figure 4.7 shows the portfolio loss density function for a reasonable rate of default and a time horizon of 5 years. We note that more systematic risk (i.e. larger v) moves more mass to the tails of the portfolio loss density function, or in other words increases the probability for no or just a few defaults as well as the probability for plenty of defaults. Comparing the shape of the NTS 1 curve with the gaussian and the NTS 2 curve leaves the first impression that the NTS 1 model incorporates more systematic risk. However, a closer look reveals that it does feature less probability mass in the right tail than the NTS 2 model. Thus, we would expect that the NTS 1 model underprices the equity tranche and the super senior tranche in comparison to the NTS 2 model. As we restrict the activity of the subordinator (i.e. increase λ) in both NTS based models this phenomenon decreases. Indeed, letting α go to 2 and λ to infinity while b equals zero all models tend to become equal.

4. Model calibration

To assess the strengths and weaknesses of the previously introduced models we will now move on and calibrate them to a real set of financial time series. We will evaluate their capabilities with respect to the goodness of fit.

4.1. Data

We use iTraxx Europe Series 7 on-the-run (March 20 to September 20, 2007) 5 year index spreads and the corresponding first five standard tranche spreads with attachement / detachment points: 0%, 3%, 6%, 9%, 12%, 22%.

The first reason for choosing this set of data was liquidity and the accompanying low bid-ask spread. iTraxx Europe series are among the most actively traded credit portfolios and synthetically replicate an equally weighted portfolio of 125 investment grade single names. The index gets rolled every six months and covers five different sectors. Beyond the rating, liquidity in their CDS trading is a selection criterion for singles names to become an index member.

The second reason is that it covers various states of the economy. This will allow us to assess a model's performance in the outburst of the crisis when market expectations and risk aversion changed remarkably. Compared to a complete in-crisis data set, we will additionally reduce the impact of frozen markets and large bid-ask spreads. Though the time series does not contain any defaults, it has been heavily affected by the outburst of the financial crisis.

Like implied volatility in equity option pricing, implied correlation in general is not constant over different tranches. In line with equity derivatives pricing, this phenomenon is called the correlation smile or skew. Yet, unlike the volatility smile / skew, the shape of implied correlation as a function of detachment points does not necessarily have any similarity with a smile or skew. In fact, depending on the underlying approach - base or compound correlation - the shape is remarkably different. Still, the message is the same. The correlation smile reflects nothing as the models incapability to reproduce market spreads using a flat correlation. Implied correlation is simply abused as another degree of freedom without any economic justification.

Thus, we will use the extent to which a model is capable to reproduce market spreads as an optimisation criterion when calibrating the models. In detail, we use a constrained non-linear optimisation routine to determine the most suitable free parameters (parameters of the copula and ρ) to reduce the *relative root mean squared error (rRMSE)*[1]:

$$rRMSE = \sqrt{\frac{1}{n}\sum_{k=1}^{n}(\frac{s_k^{model} - s_k^{market}}{s_k^{market}})^2}$$

, where the index k denotes the respective tranche in ascending order.

We have implemented the model variants listed below. Where used the base correlation approach to calculate implied correlation, as it is the market standard for most synthetic credit portfolios.

- Gaussian model: free parameter ρ.
- NTS 1 model: free parameter α, θ and ρ. β has been set to zero.
- NTS 2 model with $\beta = 0$: free parameters α, λ and ρ.
- NTS 2 model: free parameters α, λ, β and ρ.

In the Gaussian case we have directly minimized the rRMSE. In all other variants, we have determined the ρ that sets the equity tranche model spread equal to the market spread and subsequently used this ρ to calculate the rRMSE that considers all tranches. Consequently, the model's equity tranche spread will always perfectly match the corresponding market spread. Through variation of the parameters of the copula function, we then search for the parameter set that leads to the globally minimal rRMSE. Although this approach is slightly dominated by the one used for the Gaussian model (in terms of optimality) it is less computationally heavy. Indeed, as the Gaussian model turns out to be inferior to all other models examined, this should not bias the results of our analysis.

4.2. Results

The main findings are as follows:

- The NTS 2 model is the only model that reprices market spreads acceptably with a flat correlation coefficient (see figure 4.8). Thus, the implied base correlation smile nearly disappears under the NTS 2 assumption.
- Implied base correlation is significantly lower in the NTS 2 model, as a result of the endogenously stronger consideration of correlation risk.

[1] see Schmitz et al. (2010)

- In line with our previous concerns, the results suggest that the random variables Z and ϵ_i should be independently distributed.
- Allowing for values of β unequal to zero leads to a better fit of market prices.
- Under the NTS approach, changes in base correlation account for a greater proportion of the overall spread movement.

The corresponding figures and tables can be found at the end of this chapter: In table 4.1, we summarize the maximum absolute relative deviation of model vs. market spread and the parameter range of values for each model. Figure 4.8 shows the rRMSE over time and figure 4.9 to 4.12 plot the base correlation surface for the various models.

For the Gaussian model, we observe that if a flat correlation parameter is used, the mispricings are tremendous for all considered tranches. In general, the equity tranche as well as the $9\% - 12\%$ and the $12\% - 22\%$ tranches are underpriced, whereas the $3\% - 6\%$ and the $6\% - 9\%$ tranches are overpriced. In other words, the Gaussian copula provides insufficient probability mass in the tails. As a matter of fact, implied base correlation increases as a function of detachments, when we allow for varying base correlations (see figure 4.9). We explained the reasons for this before, when we demonstrated the Gaussian models inability to account for extreme joint events.

In line with our previous concerns, the NTS 1 model fails to be a sufficient alternative to the Gaussian model. In particular, when market spreads widened, the NTS 1 model is anything but better. Although the model possesses a significantly higher *lower tail dependence coefficient* it constantly underprices the senior tranche ($12\% - 22\%$). This gives evidence to our belief that idiosyncratic and systematic risk should be independently modelled. The parameters α and θ behave reasonably over time and affect that more probability mass goes to the tails of the copula function with the outburst of the crisis.

The combination of significant tail dependence and independent risk factors appears to be a suitable solution: In both configurations of the NTS 2 model, mispricings are negligible over the first 80 days. After that, when market spreads increased, the NTS 2 model started to overprice the senior tranches. Its ability to account for extreme joint events clearly distinguishes it from the previous two models. If we allow $\beta \neq 0$, the rRMSE can be reduced as this dams the underpricing of the mezzanine tranches. However, adding another free parameter increases the complexity of the optimisation problem. As a last point, we observe that all parameters behave reasonably, e.g. α decreases with the outburst of the crisis and β becomes even more negative. Hence, we have no reason to believe that the model's performance is solely based on its many degrees of freedom. It rather accounts for the various drawbacks of the Gaussian copula.

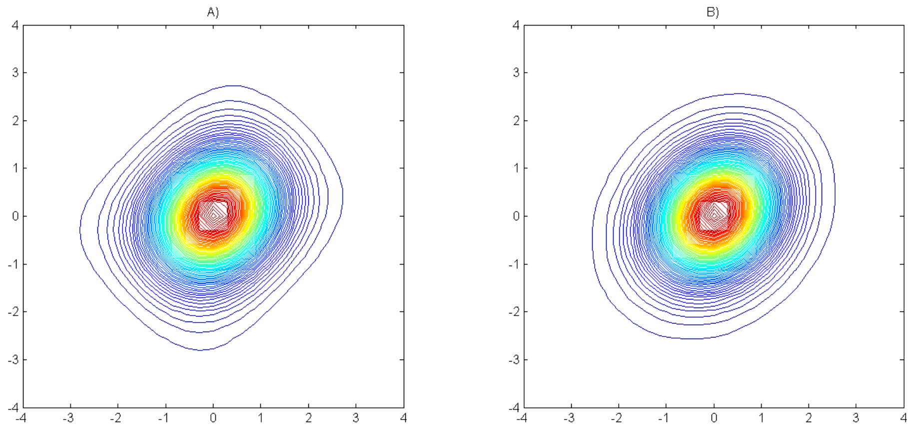

Figure 4.1.: Contour plot of bivariate stdNTS pdf ($\alpha = 1.8$, $b = -0.1$, $\lambda = 0.2$, $\rho_{1,2} = 0.2$)

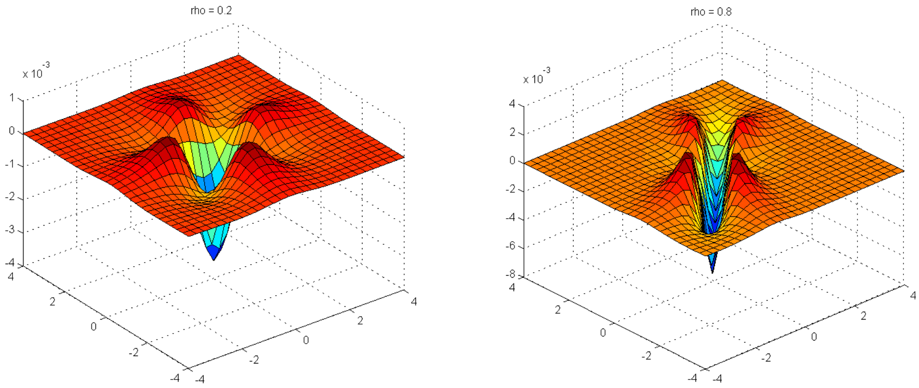

Figure 4.2.: Common subordinator vs. mutually independent subordinators, ($\alpha = 1.8$, $b = -0.1$, $\lambda = 0.2$)

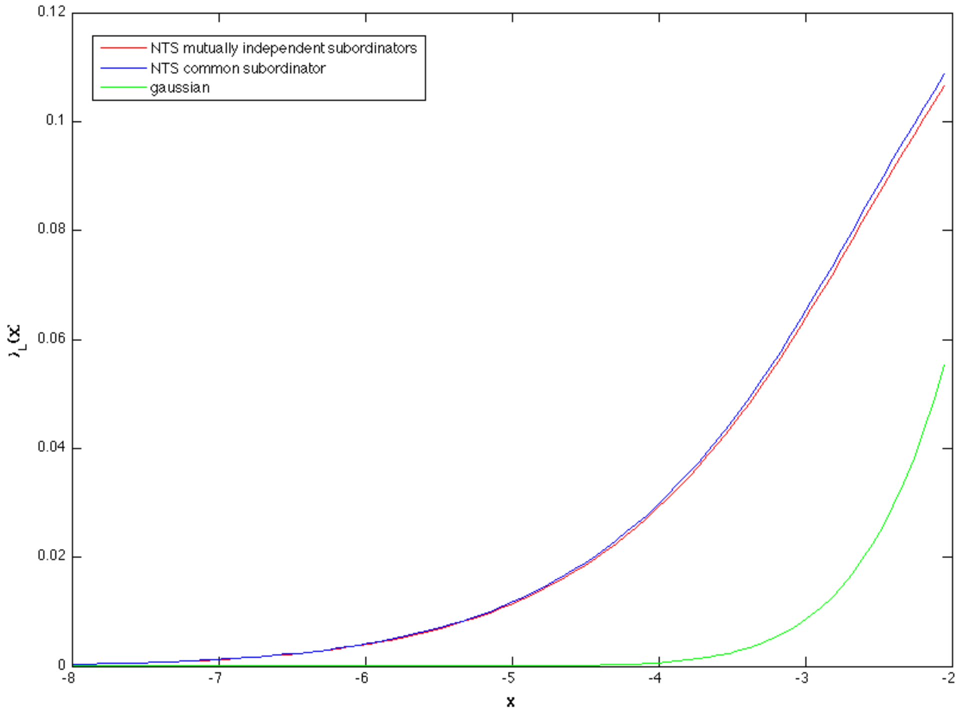

Figure 4.3.: stdNTS: lower tail dependence, ($\alpha = 1.8$, $b = -0.1$, $\lambda = 0.2$, $\rho_{1,2} = 0.2$)

Tranche	$0-3\%$	$0-3\%$	$0-3\%$	$0-3\%$	$0-3\%$
Gaussian	0.7139	1.4391	0.6214	0.9385	0.9967
NTS 1 ($\beta = 0$)	0	1.4864	1.1495	0.7939	0.5098
	$\alpha \in [1.7475, 1.9739]$		$\theta \in [1e-04, 0.0040]$		
NTS 2 ($\beta = 0$)	0	0.2953	0.2038	0.3502	0.7752
	$\alpha \in [1.01, 1.9263]$		$\lambda \in [3.69e-04, 0.8212]$		
NTS 2	0	0.2135	0.1069	0.2036	0.5515
	$\alpha \in [1.0008, 1.99]$		$\lambda \in [8.26e-04, 0.6368]$		$b \in [-5.2156, 0.5483]$

Table 4.1.: Maximum absolute relative deviation of model vs. market spread and parameter range of values

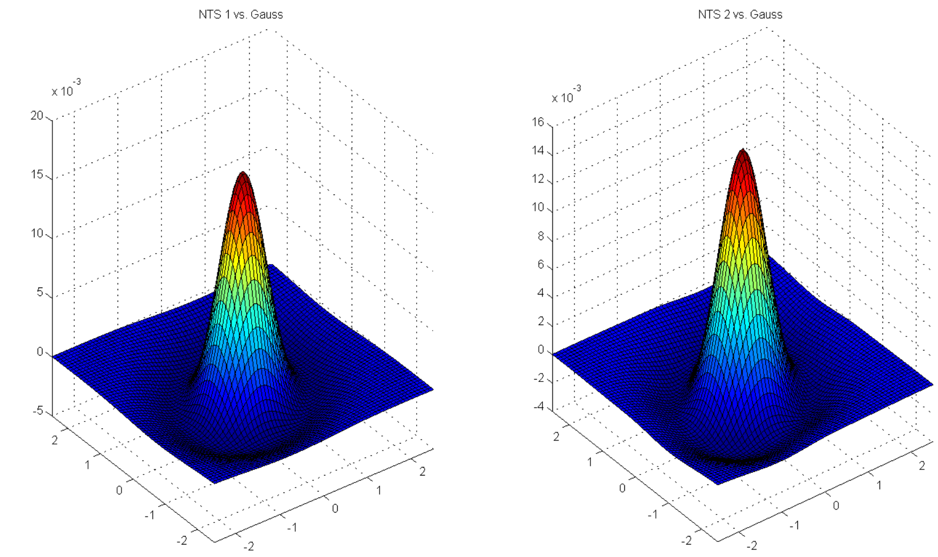

Figure 4.4.: Comparison of joint default probability distributions (NTS - Gauss), NTS marginal distribution ($\alpha = 1.6$, $b = 0$, $\lambda = 0.2$, $\rho = 0.3$)

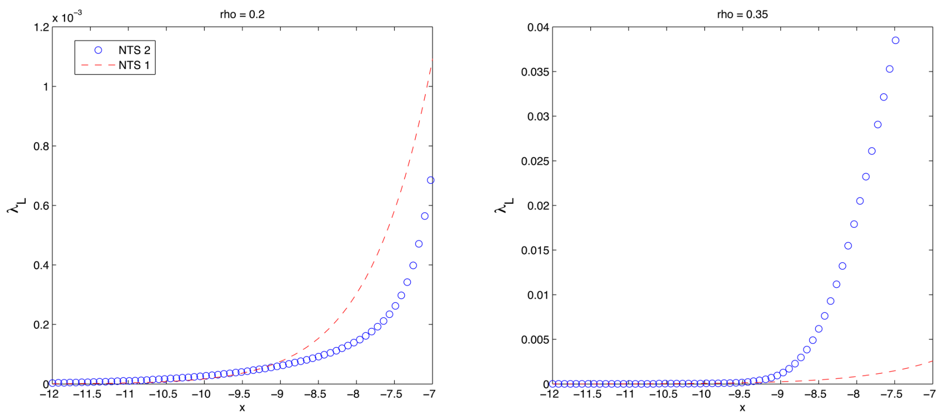

Figure 4.5.: Comparison of the *lower tail dependence coefficient* λ_L, NTS marginal distribution ($\alpha = 1.6$, $b = 0$, $\lambda = 0.2$)

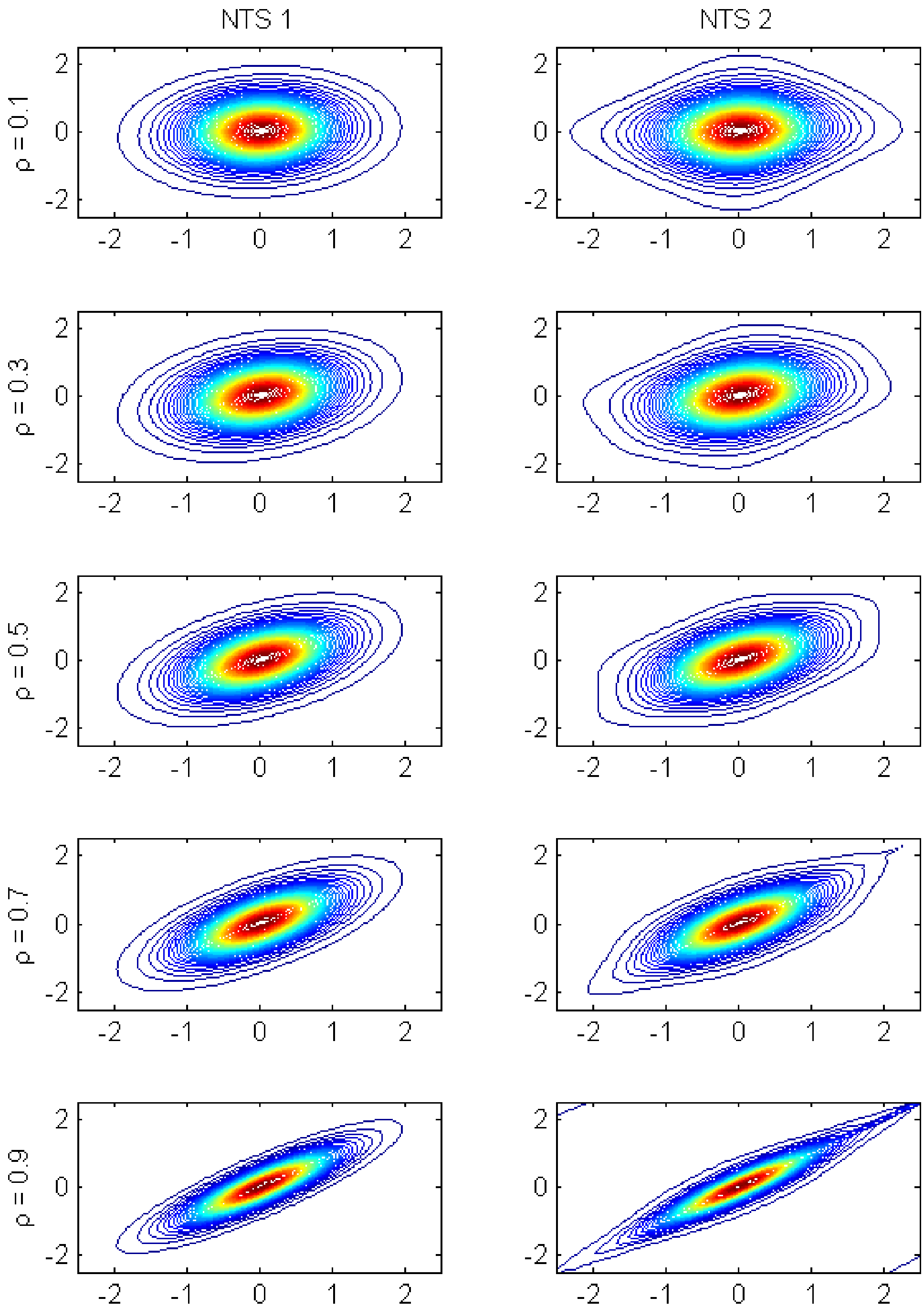

Figure 4.6.: Contour plots of NTS 1 and NTS 2 copula functions, NTS marginal distribution ($\alpha = 1.6$, $b = -0.05$, $\lambda = 0.2$)

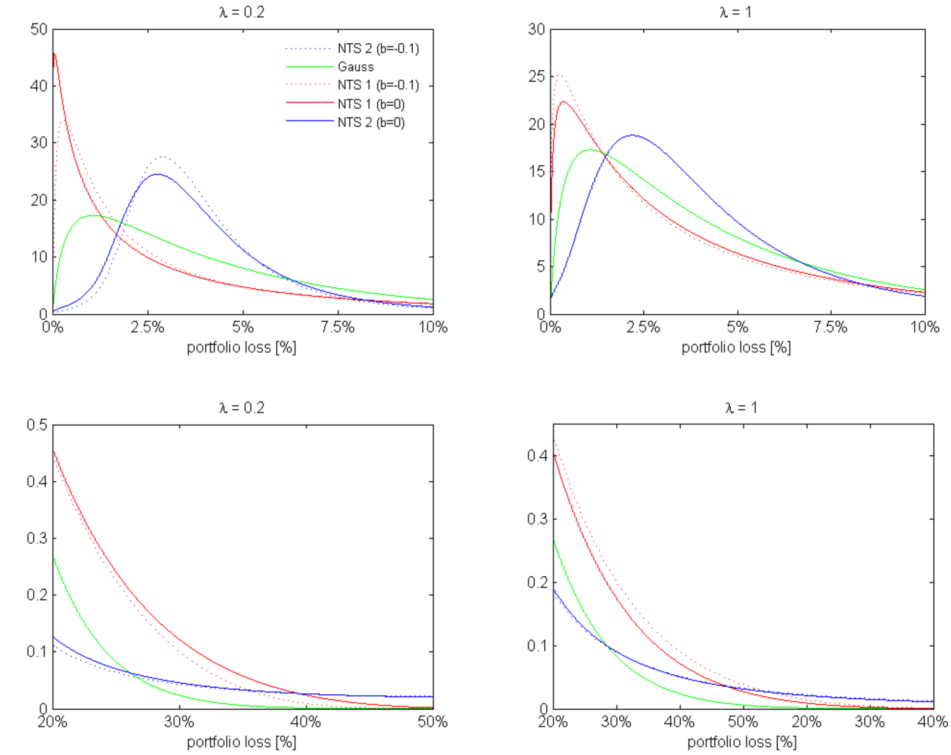

Figure 4.7.: Comparison of LHP portfolio loss density functions, NTS marginal distribution ($\alpha = 1.6$, $\rho = 0.2$)

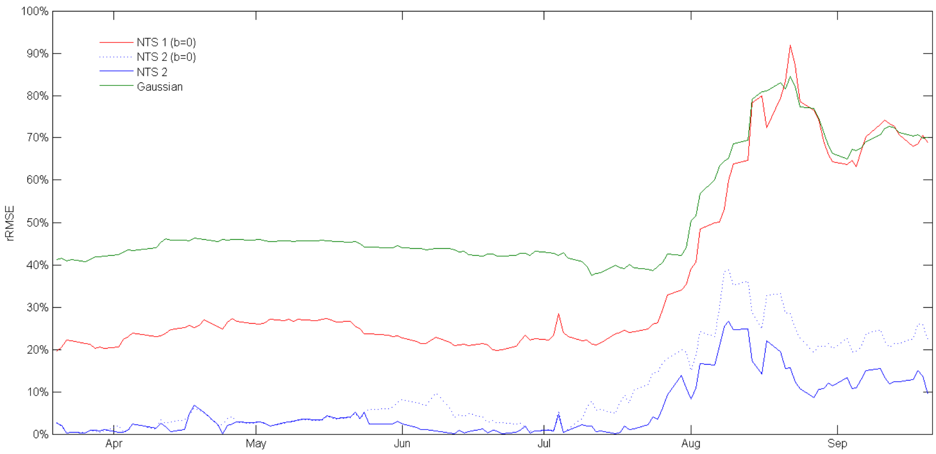

Figure 4.8.: Relative root mean squared error

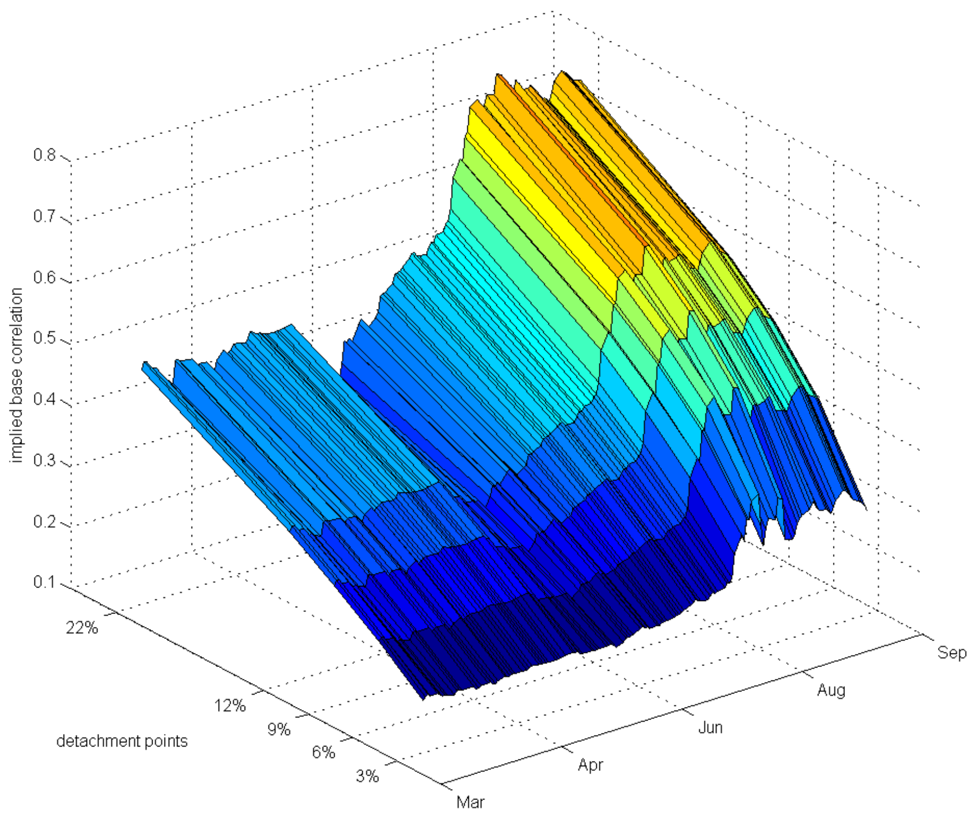

Figure 4.9.: Gaussian copula model - implied base correlation surface

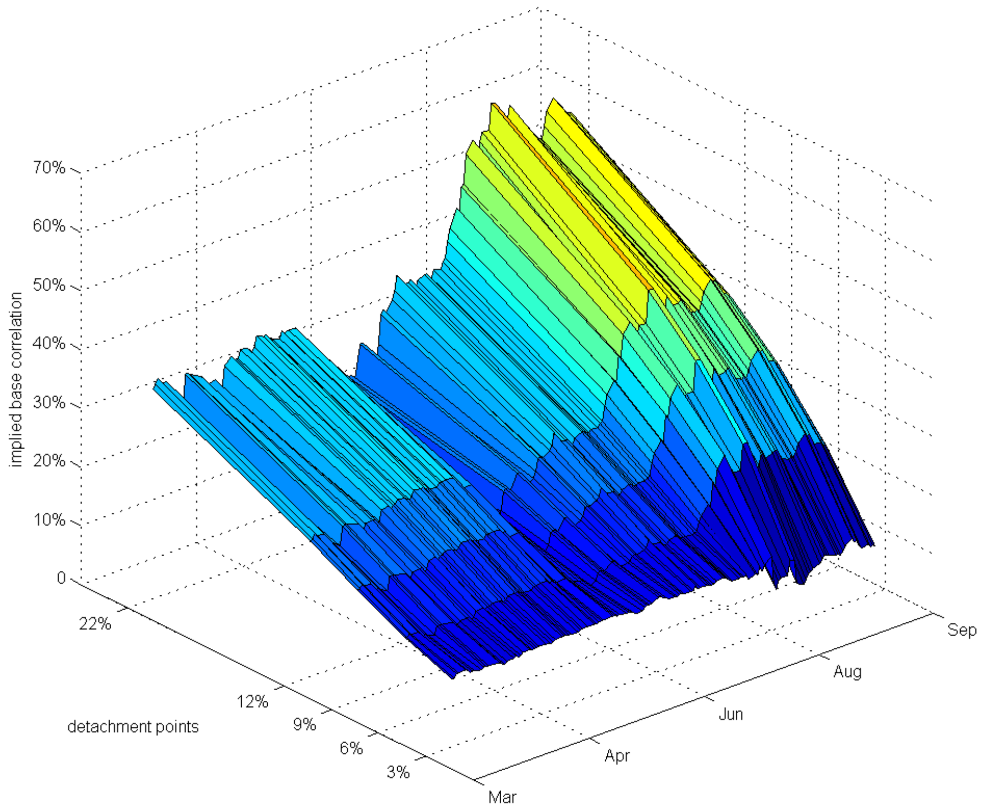

Figure 4.10.: NTS 1 copula model - implied base correlation surface

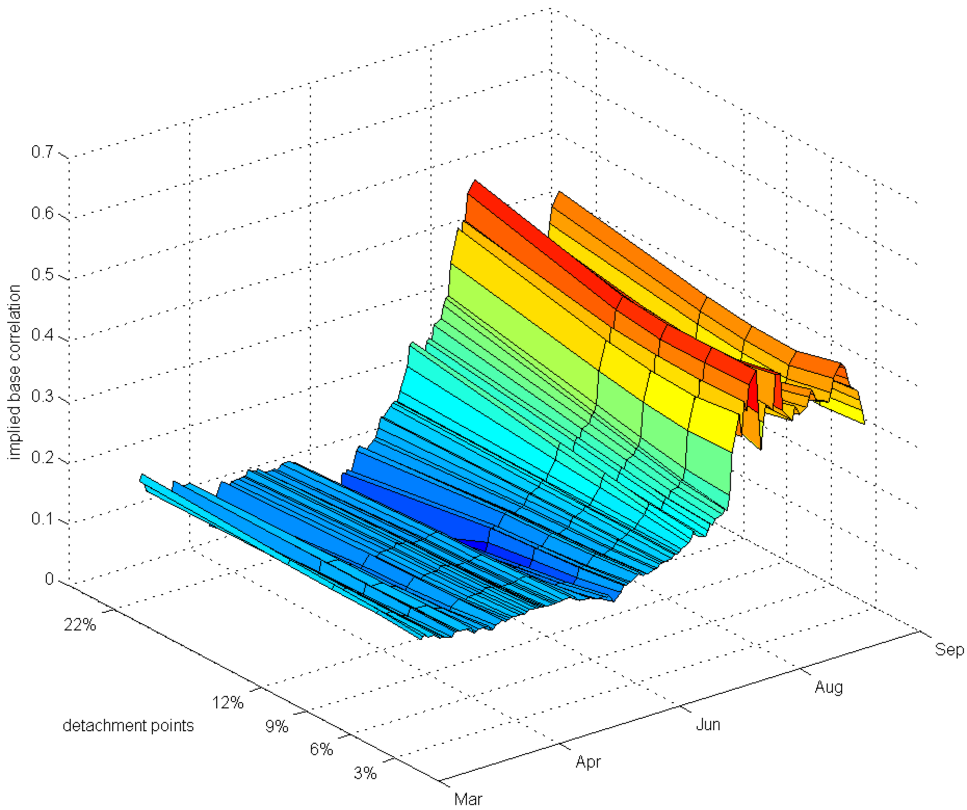

Figure 4.11.: NTS 2 copula model ($b = 0$) - implied base correlation surface

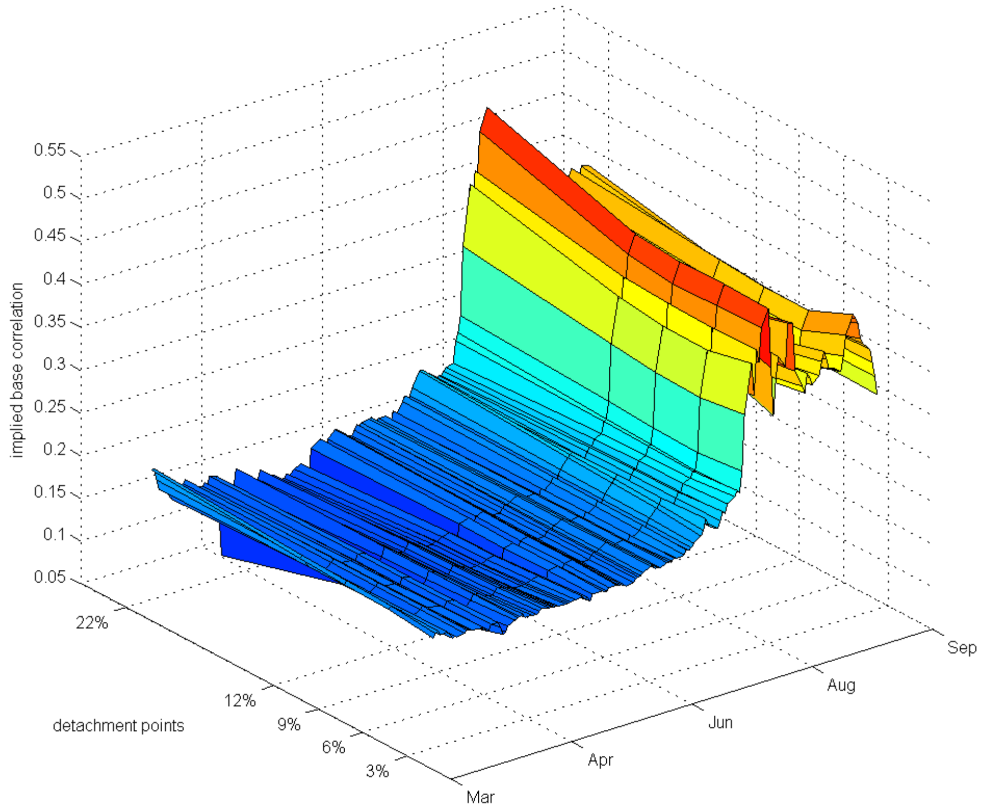

Figure 4.12.: NTS 2 copula model - implied base correlation surface

Appendix

A. Appendix A: CTS subordinator - effect of parameters

We illustrate the behaviour of the shape of the density function for various parameterizations in Figure A.1. Table A.1 shows the effects on the first four standardized moments. Please be aware that the distribution is standardized such that $E[T] = 1$ holds.

Looking at the figures we observe that irrespective of what parameter we increase the density function peaks higher and more probability mass moves to the righthand side of the plot while the density function starts to decrease more rapidly. In addition, the variance decreases in α and θ. However, the effect of α and θ on skewness and kurtosis are adverse. While a higher α leads to more skewness and excess kurtosis the opposite is true for θ. We note that the magnitude ot the effect of a small change in parameter on skewness and kurtosis is more intense for large values of α as well as for small values of θ.

α	θ	Variance	Skewness	Excess Kurtosis
1.6	0.2	1	6	66
1.6	0.02	10	18.97	666
1.6	2	1.90	0.1	6.6
0.8	0.2	3	4.62	34.67
1.9	0.2	0.25	10.5	215.25

Table A.2.: CTS subordinator - effect of parameters for t = 1

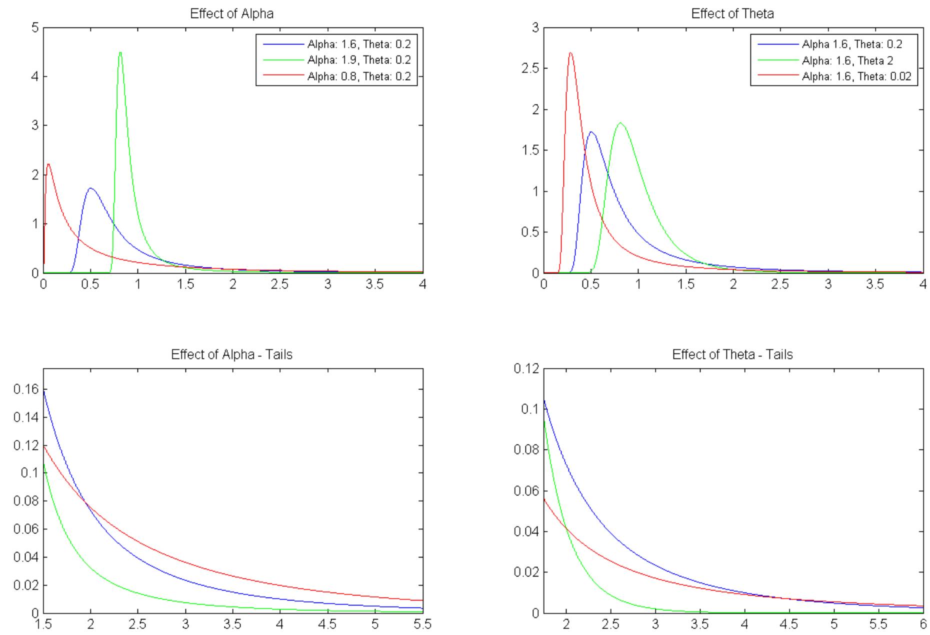

Figure A.1.: CTS subordinator - effect of parameters for t=1

B. Appendix B: standard NTS - effect of parameters

Analogous to Appendix A, we discuss the effect of a change in parameter on the shape of the density function as well as on the skewness and the kurtosis of the distribution.

We first remind ourselves that the Normal Tempered Stable distribution can be obtained by tempering the Lévy measure of a symmetric α-stable distribution. While α defines the Lévy measure of the α-stable distribution θ steers the tempering. Thus we conclude that a lower α leads to an increase in the probability for large jumps and as θ increases those jumps get more heavily tempered. Another common way to illustrate the effect of θ is to consider it as a *rate of decay.* As θ increases, i.e. the tails decay faster the density function simply has to peak higher and provide more mass in the tails as the distribution is standarized. This explains why the kurtosis falls in θ and α. Loosely speaking, the standardized gaussian distribution is a limiting case of the normal tempered stable distribution as α goes to 2. Though as $\alpha \in (0, 2)$ this case can never be reached and both distributions will differ from each other as follows: In generel, the stdNTS distribution will peak higher and provide more mass in the tails then the standardized gaussian distribution. Again, as we're talking about standardized distribution, i.e. the distribution is scaled to unit variance, the stdNTS exhibits less mass on both legs (see Figure B.2).

Finally, β is related to the distribution's skewness. A negative value of β leads to a negatively skewed density function (et vice versa) while in case $\beta = 0$ the density function remains symmetric.

α	θ	β	Skewness	Excess Kurtosis
1.6	0.2	0	0	3
1.6	0.2	-0.1	0.2640	3.1625
1.6	0.2	0.1	-0.26400	3.4185
1.6	0.02	0	0	30
1.6	2	0	0	0.3
0.8	0.2	0	0	9
1.9	0.2	0	0	0.75

Table B.3.: standard NTS - effect of parameters

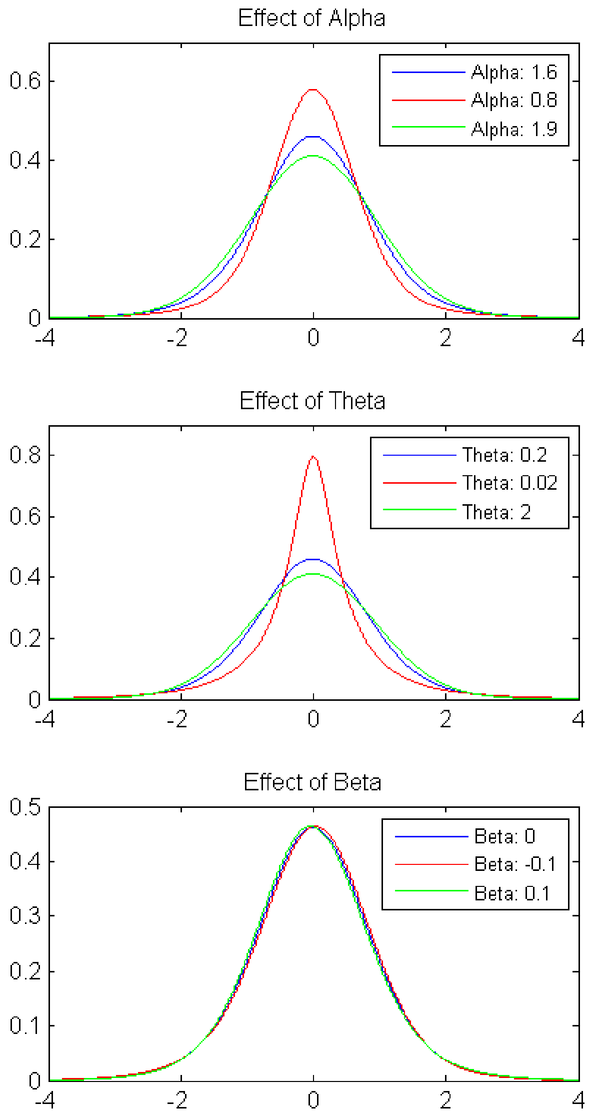

Figure B.2.: standard NTS - effect of parameters, unless otherwise specified $\alpha = 1.6$, $\theta = 0.2$, $\beta = 0$.

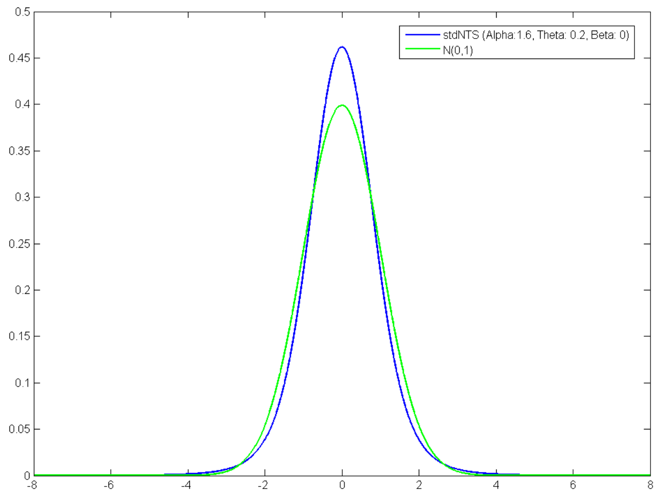

Figure B.3.: standard NTS vs. standard Gauss

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