Computing VaR and AVaR of Skewed-T Distribution

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Abstract

We consider the skewed-T distribution defined as a normal mixture with inverse gamma distribution. Analytical formulas for its value-at-risk, VaR quantile, and average value-at-risk, AVaR conditional mean are derived. High-accuracy approximations are developed and numerically tested.

Keywords: *skewed-T distribution, asymmetric, value-at-risk, VaR, AVaR*

1 Introduction

The skewed-T distribution is a popular choice for modeling financial time series of asset returns. The VaR quantile and the average VaR quantile, i.e. AVaR, of

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those returns are usually estimated from a large sample of observations. If such large sample is not available, as in a case when only short history of returns is present, then we need a reliable way for assessing the magnitude of the VaR and AVaR risk measures. Analytical formulas might help in this case and let thorough analysis been performed on the risk measures by varying the distribution parameters and assessing different risk levels.

We note that asymmetric distributions can be defined in different ways from their symmetric counterparts. For one such case of skewed-T distribution we see analytical formulas for VaR and AVaR derived in [1] (AVaR is denoted CVaR in [1]). In this case the skewed Student-t density function was proposed by Hansen (1994) in [2]. The density is an extension of the conventional symmetric Student-t distribution. The asymmetry is introduced by weighting differently, multiplying by different weights, the negative and the positive values of the symmetric Student-t distribution.

We consider skewed-T distribution defined as a normal mixture with inverse gamma distribution (e.g. see [3] for details). Such skewed-T random variable, X, is defined by

$$X = \mu + \gamma W + Z\sqrt{W},$$

where $W \sim Ig(\nu/2, \nu/2)$, i.e. W is inverse gamma random variable, Z is multivariate normal random variable $Z \sim N_d(0, \Sigma)$, and W, Z are independent. In the paper we present the analytical formulas for α-level VaR(X) and AVaR(X) risk measures. We denote the d-dimensional distribution by $X \sim t_d(\nu, \mu, \Sigma, \gamma)$ where ν stands for degrees of freedom, $\nu \geq 4$, μ is a location parameter, and Σ is d-by-d covariance matrix. Finally, the sign of γ controls the distribution asymmetry: positive for skewed to the right, having fat right tail of asset returns, and vice-versa, negative for skewed to the left; except for the case $\nu = 5$.

In the paper we specifically consider the cases for $\gamma \neq 0$, that is, the cases with "true" asymmetry in X. Nevertheless, we note that for small γ's (of order 10^{-6}) our formulas numerically converge to the symmetric, conventional Student-t, AVaR formula

$$AVaR_\alpha(X) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{\sqrt{\nu}}{(\nu-1)\alpha\sqrt{\pi}} \left(1 + \frac{(VaR_\alpha(X))^2}{\nu}\right)^{\frac{1-\nu}{2}}$$

for $\nu > 1$.

The paper is organized as follows: in next Section 2 we state the classical definitions of VaR and AVaR. Then we elaborate on the form of the skewed-T pdf needed for development of the analytical formulas. An integral representation of the Bessel function (involved in the pdf) is utilized. The analytical formulas for VaR and AVaR are stated in two theorems. Their properties are discussed and one related proposition is stated. The proofs of the theorems and the proposition are presented in the appendix with thorough details. Two corollaries state approximation versions of the theorems statements. The corollaries results are very useful in numerical implementations. Theire proofs are in the appendix as well. In Section 3 we discuss some issues arising in numerical implementations of the developed formulas. The issues are resolved via the Bessel function asymptotic forms and via locating the spike-like unimodal peak of the quadrature integrand function. The paper is concluded in Section 4.

2 VaR and AVaR for skewed-T distribution

The definition of VaR calls for a confidence level $\alpha \in (0,1)$. Then the VaR of portfolio return at confidence level α is defined as the smallest number x_0 such that the probability that the loss X exceeds x_0 is not larger than $(1-\alpha)$. That is, in general

$$\begin{aligned} VaR_\alpha(X) &= \inf \{x_0 : P(X > x_0) \le 1-\alpha\} \\ &= \inf \{x_0 : F_X(x_0) \ge \alpha\} \\ &= F_X^{-1}(\alpha) \end{aligned}$$

where $F_X(\cdot)$ is the cdf (cumulative distribution function) of X, F_X^{-1} is the inverse function of F_X provided one exists, and the last equality holds for continuous distributions. In probabilistic terms VaR is the α-quantile of the loss distribution. If we consider the random variable X for modeling the asset returns then $-X$ models the asset losses. Here we will not distinguish between the two, but we will derive analytical formulas for both tails of the skewed-T distribution, that is, formulas for smaller and larger α values for the VaR quantile.

If we let the random variable X denote portfolio loss then the definition of the α level AVaR is given by the following conditional expectation

$$\begin{aligned} AVaR(X) &= E\left[X | X \ge VaR_\alpha(X)\right] \\ &= \frac{1}{1-\alpha}\int_{X \ge x_0} x f(x) dx \end{aligned}$$

that is, the α level average value-at-risk $AVaR(X)$ is the average loss larger than the α level quantile loss $VaR_\alpha(X)$. Similarly to the VaR case, we will derive AVaR results for both distribution tails combined with the two cases for the sign of the asymmetry parameter γ which controls the fatness of the distribution tails.

For our analytical results we need the probability density function, $f(x)$, of the skewed-T random variable X, which is given by

$$f(x) = \frac{2^{1-(\nu+d)/2}}{\Gamma(\frac{\nu}{2})(\pi\nu)^{d/2}\left|\Sigma\right|^{1/2}} * \frac{\exp((x-\mu)'\Sigma^{-1}\gamma)}{\left(1 + \frac{(x-\mu)'\Sigma^{-1}(x-\mu)}{\nu}\right)^{(\nu+d)/2}}$$

$$* \frac{K_{(\nu+d)/2}\left(\sqrt{(\nu + (x-\mu)'\Sigma^{-1}(x-\mu))\gamma'\Sigma^{-1}\gamma}\right)}{\left(\sqrt{(\nu + (x-\mu)'\Sigma^{-1}(x-\mu))\gamma'\Sigma^{-1}\gamma}\right)^{-(\nu+d)/2}}$$

where $K_\lambda(\cdot)$ is the modified Bessel function of the third kind with index λ (also know as modified Hankel function or Macdonald function). For details see, for example, [4] and [5]. The skewed-T distribution is in the class of generalized hyperbolic distributions. When we model asset returns with this distribution then the resulting portfolio return can be seen as a random variable which is a linear combination of skewed-T returns. Because of a property of the generalized hyperbolic distributions such linear combination has a generalized hyperbolic distribution as well (see Corollary 2.2.4 in [8]).

For assessing the VaR and AVaR of a single asset we consider its return as univariate, $d = 1$, skewed-T random variable. The corresponding univariate probability density (pdf) function is

$$f(x) = \frac{\nu^{\lambda}\gamma^{2\lambda}}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}2^{2\lambda}} \int_0^{\infty} t^{-\lambda-1} e^{-t-\frac{\nu\gamma^2}{4t}} e^{\gamma x - \frac{(\gamma x)^2}{4t}} dt$$

where, for convenience, we set $\mu = 0$, $\Sigma \equiv \sigma^2 = 1$, and $\lambda = (\nu + 1)/2$. That is, we consider X normalized by the standard transformation

$$\frac{X - \mu}{\sigma} = \frac{\gamma}{\sigma} W + N(0, 1)\sqrt{W}.$$

where $N(0, 1)$ stands for a random variable from the standard normal distribution. For numerical implementations we note that the skewness controlling parameter γ of the normalized skewed-T random variable $\frac{X-\mu}{\sigma}$ actually becomes $\frac{\gamma}{\sigma}$ when the above normalized pdf $f(x)$ is utilized.

In the last form of the pdf, $f(x)$, we applied the following integral representation of the Bessel function (with $y = \sqrt{(\nu + x^2)\gamma^2}$)

$$K_{\lambda}(y) = \frac{1}{2} \left(\frac{y}{2}\right)^{\lambda} \int_0^{\infty} t^{-\lambda-1} e^{-t-\frac{y^2}{4t}} dt.$$

Among other places this representation can be seen in [6]. We note that the skewed-T distribution is also popular under the name asymmetric Laplace distribution, for example, in statistical applications in medical research: see [7] where the inverse gamma random variable W is replaced with a special case of its reciprocal values.

2.1 VaR formula for skewed-T distribution

In this section we develop a method for computing $VaR(X)$, i.e., the $1 - \alpha$ quantile of a skewed-T random variable X. We look for a formula and/or numerical procedure yielding $x_0 = \mathrm{VaR}_{\alpha}$ which is such that

$$1 - \alpha = \int_{x_0}^{\infty} f(x)dx$$

where $f(x)$ is the univariate density which is also normalized with the notations we introduced.

For a given skewed-T random variable we know the sign of the skewness controlling parameter γ yielding heavier right or left distribution tail. Similarly, we have to know the sign of x_0, the α level $VaR(X)$, in order to distinguish between the two distribution tails (that is, we have to know whether we look for the VaR quantile for a "smaller" or for a "larger" α value). We so, first, compute the above integral on the interval $[0, \infty]$, and we set

$$1 - \alpha_0 = \int_0^{\infty} f(x)dx.$$

Hence, if the given VaR level α is such that $\alpha < \alpha_0$ then we look for negative x_0, otherwise, we look for positive x_0. This naturally leads to two cases in the $x_0 = \mathrm{VaR}_\alpha$ calculation depending on whether the specified VaR level α is greater than or less than the α_0 value we set. These two cases will have to be combined with the other two cases coming from the sign of the skewness controlling parameter γ. This naturally leads to total of four cases which we describe and study in this section and in the next section with respect to the VaR and AVaR formulas.

The formula for computing α_0 comes as a corollary of the following theorem.

Theorem 1. *The VaR formula for skewed-T random variable, that is, the value of $x_0 = VaR_\alpha$ is coming as the unique zero of the following equation (provided $\gamma > 0$)*

$$g(x_0) = -\alpha + \frac{2C\sqrt{\pi}}{\gamma} \int_0^\infty t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right) dt = 0.$$

For negative skewness the value of $x_0 = VaR_\alpha$ is coming as the unique zero of the next equation (i.e., provided $\gamma < 0$)

$$g(x_0) = 1 - \alpha + \frac{2C\sqrt{\pi}}{\gamma} \int_0^\infty t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right) dt = 0,$$

In both cases the zero, x_0 of g, is sought on the interval $(-\infty, 0]$ provided $\alpha < \alpha_0$ or, on the interval $[0, +\infty)$ provided $\alpha > \alpha_0$.

Proof. For the case $\gamma > 0$ see the Appendix. □

In the theorem statement the $\Phi(\cdot)$ stands for the standard normal cdf, the constant

$$C = \frac{\nu^\lambda \gamma^{2\lambda}}{\sqrt{\pi\nu}\Gamma(\frac{\nu}{2})2^{2\lambda}}$$

depends on the skewed-T distribution parameters: degrees of freedom ν, $\lambda = (\nu + 1)/2$, and γ is the skewness controlling parameter.

We note that $g(\cdot)$ is an increasing function of x_0 from $(-\alpha)$ to $1 - \alpha$ as x_0 ranges from minus to plus infinity. Hence, the unique zero of g can be found by any numerical routine, for example, by one like a bi-sectional search.

The integrals on infinite intervals in the theorem have to be computed numerically. Also the search for the zero x_0 of g has to be performed on infinite intervals. We so derive approximations of the theorem statement where the infinite intervals are replaced with finite intervals. We prove that the accumulated numerical error is bounded by 10^{-9} when the infinite intervals are replaced by finite intervals.

Corollary 1. *The integral on infinite interval in Theorem 1 can be replaced with the following integral on a finite interval*

$$\int_0^{t_0(x_0)} t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right) dt$$

where

$$t_0(x) = \left(\frac{3\sqrt{2} + \sqrt{18 + 2\gamma x}}{2}\right)^2$$

The approximation error $R(t_0(x_0))$ *brought in*

$$g(x_0) = -\alpha + \int_0^{t_0(x_0)} t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right) dt + R(t_0(x_0))$$

for $\gamma > 0$*, and for* $\gamma < 0$ *as well, is less than* 10^{-9}.

Furthermore, the search for the zero x_0 *of* g *is performed on the following intervals*

$$\begin{aligned} [-9/\gamma, 0] & \quad \textit{if} \quad \alpha < \alpha_0, \gamma > 0, \\ [0, +\infty) & \quad \textit{if} \quad \alpha > \alpha_0, \gamma > 0, \\ (-\infty, 0] & \quad \textit{if} \quad \alpha < \alpha_0, \gamma < 0, \\ [0, -9/\gamma] & \quad \textit{if} \quad \alpha > \alpha_0, \gamma < 0, \end{aligned}$$

Proof. For the case $\gamma > 0$ see the Appendix. □

Getting rid of the error term $R(t_0(x_0))$ still preserves the increasing nature of $g(x_0)$. Hence, the search for the unique zero x_0 is fine with the approximation we propose in Corollary 1. The value of α_0 which specifies the sign of x_0 comes as corollary from the above approximation after substituting the later with zero.

Corollary 2. *The formula for* α_0 *is*

$$\begin{aligned} \alpha_0 &= \frac{2C\sqrt{\pi}}{\gamma} \int_0^{18} t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi(-\sqrt{2t}) dt, \qquad \textit{if } \gamma > 0, \\ \alpha_0 &= 1 + \frac{2C\sqrt{\pi}}{\gamma} \int_0^{18} t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi(-\sqrt{2t}) dt, \ \textit{if } \gamma < 0, \end{aligned}$$

Proof. By substitution $x_0 = 0$. □

The proofs of the theorem and the first corollary in the appendix are presented for the case $\gamma > 0$. The formulas for $g(x_0)$ when $\gamma < 0$ are derived from their $\gamma > 0$ counterparts by substituting X with $-X$, γ with $-\gamma$, and the VaR level α of X with $1 - \alpha$ which is the VaR level α for $-X$.

In both cases, for positive and negative skewness, when the zero x_0 of $g(\cdot)$ has to be sought on infinite interval

$$\begin{aligned} x_0 \in (-\infty, 0] & \text{ provided } \alpha < \alpha_0 \text{ and } \gamma < 0, \\ x_0 \in [0, +\infty) & \text{ provided } \alpha > \alpha_0 \text{ and } \gamma > 0, \end{aligned}$$

we perform some additional analysis which let us do the zero search on a finite interval. Note that these are the cases of the heavy tail in the skewed-T distribution. For the case $x_0 \in [0, +\infty)$ let us assume for example that we are looking for Value-at-Risk, $x_0 = \mathrm{VaR}_\alpha$, at confidence level α less than, say, 99.99%. Hence, $1 - \alpha \geq 0.0001$, or in general $1 - \alpha \geq \epsilon$ where the ϵ is a small positive number which we can specify in advance. That is, if we choose, for example, $\epsilon = 0.0002$ then we will be able to do the search for the zero x_0 of g on a finite interval but, the user specified VaR confidence level must not be greater than 99.98%. Thus, for a chosen small positive number ϵ, the $-/+$ infinity in the above two intervals can be replaced with $-/+$ "big" number M depending on ϵ.

Proposition 1. *The infinite intervals for the search of the unique α level $x_0 = VaR_\alpha$ can be replaced by finite intervals such that the $-/+$ infinity in Corollary 1 is replaced with $-/+M$ given by*

$$M = \frac{2d + 3\sqrt{8d}}{\mp\gamma}$$

where

$$d = \frac{\nu\gamma^2}{4}\left[\epsilon\Gamma\left(\frac{\nu+2}{2}\right)\right]^{-2/\nu}$$

and ϵ is an arbitrary small positive number specified in advance.

Proof. See the Appendix □

Here ν and γ are the skewed-T distribution parameters, and $\Gamma(\cdot)$ is the Gamma function. Technically, in the case $x_0 \in [0, +\infty)$, we have that $g(0) = -\alpha + \alpha_0 < 0$, while $g(M) \geq 1 - \alpha - \epsilon > 0$ and $g(+\infty) = 1 - \alpha > 0$. So, the interval $[0, +\infty)$ for the zero x_0 is replaced with $[0, M]$ provided that ϵ is chosen such that $\epsilon < 1 - \alpha$. Note, M depends on ϵ.

Thorough proof for the M formula in the first case, $x_0 \in (-\infty, 0]$, is presented in the Appendix.

2.2 AVaR formula for skewed-T distribution

The AVaR (average VaR, also known as conditional value-at-risk CVaR, or ETL, i.e. expected tail loss) is defined as the expectation of a distribution tail. As we already discussed it with respect to the VaR formula, for the skewed-T distribution we distinguish four cases depending on whether we are interested in computing the expectation of the left or the right distribution tail and, on the sign of the skewness controlling parameter γ. As it is also discussed, the interest in computing the conditional expectation of the left or the right distribution tail might depend on whether the distribution is utilized for modeling asset returns or asset losses. We so study all possible four cases for the conditional tail expectations.

Case 1: $x_0 \geq 0, \gamma > 0$,

$$AVaR_1 = E[X|X > x_0] = \frac{1}{P(X > x_0)}\int_{x_0}^{\infty} xf(x)dx$$

Case 2: $x_0 \leq 0, \gamma > 0$,

$$AVaR_2 = E[X|X < x_0] = \frac{1}{P(X < x_0)}\int_{-\infty}^{x_0} xf(x)dx$$

Case 3: $x_0 \geq 0, \gamma < 0$,

$$AVaR_3 = E[X|X > x_0] = \frac{1}{P(X > x_0)}\int_{x_0}^{\infty} xf(x)dx$$

Case 4: $x_0 \leq 0, \gamma < 0$,

$$AVaR_4 = E[X|X < x_0] = \frac{1}{P(X < x_0)}\int_{-\infty}^{x_0} xf(x)dx$$

The alpha level AVaR depends on x_0 which stands for the alpha level VaR, that is, $x_0 = \mathrm{VaR}_\alpha$ is the $1-\alpha$ quantile of the distribution. The x_0 is such that

$$1-\alpha = \int_{x_0}^{\infty} f(x)dx$$

The VaR_α is studied in the previous section.

Along with the notations we have so far, like C and $t_0(x)$, here we introduce two additional notations

$$\beta = \sqrt{\nu\gamma^2 + (\gamma x_0)^2} \quad \text{and} \quad h_0 = \frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}$$

and, an expression which repeatedly appears in the final AVaR formulas

$$KI = \left[K_{\lambda-1}(\beta) \left(\frac{2}{\beta}\right)^{\lambda-1} e^{\gamma x_0} - \sqrt{\pi} \int_0^{\infty} t^{-\lambda+1/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi(h_0) dt \right]$$

which is the difference between the modified Bessel function of the third kind with index $(\lambda-1)$ and a finite integral. Moreover,

$$KI = K_{\lambda-1}(\beta) \left(\frac{2}{\beta}\right)^{\lambda-1} e^{\gamma x_0}, \quad \text{if} \quad 9 + \gamma x_0 < 0$$

that is, the integral vanishes (has a value of order 10^{-9}) for γ and x_0 such that their values satisfy the last inequality.

Theorem 2. *The AVaR formula for skewed-T random variable, in each one of the four cases we describe, is*

$$\begin{aligned}
AVaR_1 &= \frac{\gamma\nu}{(1-\alpha)(\nu-2)} + \frac{4C}{(1-\alpha)\gamma^2} KI \\
AVaR_2 &= \frac{-4C}{\alpha\gamma^2} KI \\
AVaR_3 &= \frac{4C}{(1-\alpha)\gamma^2} KI \\
AVaR_4 &= \frac{\gamma\nu}{\alpha(\nu-2)} + \frac{-4C}{\alpha\gamma^2} KI
\end{aligned}$$

Proof. See the Appendix for all details about the proof of the $AVaR_1$ formula. The $AVaR_2$ formula is derived as a complement for $AVaR_1$ to the mean, $E(X) = (\gamma\nu)/(\nu-2)$, of the normalized skewed-T random variable X. The other two formulas, $AVaR_3$ and $AVaR_4$, come from the first two after the substitution of X with $-X$, the substitution of the skewness parameter γ with $-\gamma$, and the substitution of $\alpha = \alpha_X$ with $(1-\alpha) = \alpha_{(-X)}$. □

We note that the AVaR calculation for the skewed-T distribution requires numerical integration on infinite interval. Similarly to the previous section we approximate it with integration on a finite interval. Next we state this result.

Corollary 3. *The integral on infinite interval in the KI expression in Theorem 2 can be replaced with the following integral on a finite interval*

$$\int_0^{t_0(x_0)} t^{-\lambda+1/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi(h_0) dt$$

The approximation error is lees than

$$\frac{\nu\gamma}{(1-\alpha)(\nu-2)} 10^{-9}.$$

Proof. See the Appendix for all details about the proof of the $AVaR_1$ approximation. □

Theorems 1 and 2 provide the analytical formulas for VaR and AVaR of the skewed-T distribution defined as a normal mixture with inverse Gamma distribution. The approximation versions of the formulas presented in Corollaries 1 and 3 allow to carry on numerical tests of those formulas. The results are presented in the next section.

3 About issues in numerical implementations

In this section we present our findings when the analytical VaR and AVaR formulas are tested in numerical experiments. We, first, generate ten million variates from the skewed-T distribution. This let us achieve good sample estimates for VaR and AVaR quantities at different confidence levels α. We vary the confidence level from one to ninety nine percent. Then we compare the estimated quantities with their analytical counterparts. The percent relative error $100 * |(sampleEstimate - analyticalResult)/sampleEstimate|$ stays below one percent. But, when we begin vary the distribution parameters significantly then we discover that some additional theoretical work must be done.

Two important issues deserve our attention. First, the accuracy of the numerical integration, and second, the asymptotic behavior of the modified Bessel function of the third kind. The later is well studied in the scientific literature. We so only point out the way we utilize this asymptotic behavior for achieving high numerical accuracy.

About the first issue, the accuracy in the numerical integration, after extensive numerical testing we note that the integrand function, that is, the function defining the quadratures has a spike-like shape which can easily destroy the accuracy in the numerical integration. Finding the exact point at which the spike appears requires additional amount of numerical calculations or theoretical analysis. So, we basically determine that the spike must appear near the point $\gamma^2/2$. This fact is stated and proved in the next proposition.

Proposition 2. *The integrand function in the quadratures in Theorems 1 and 2, respectively, in Corollaries 1 and 3*

$$u(t) = t^{-\theta} e^{-\frac{\nu\gamma^2}{4t}} \Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right)$$

where $\theta = (\nu+2)/2$, or $\theta = \lambda - 1/2 = \nu/2$, has maximum for $t > 0$ in a neighborhood of $t = \gamma^2/2$.

Proof. We use the notation h_0 introduced in the previous section. The first derivative of $u(t)$ becomes

$$u'(t) = t^{-\theta-2} e^{-\frac{\nu\gamma^2}{4t}} \left[\left(\frac{\nu\gamma^2}{4} - \theta t \right) \Phi(h_0) - (\gamma x_0 + 2t) \frac{\sqrt{t}}{2\sqrt{2}} \Phi'(h_0) \right]$$

The first expression multiplying the cdf $\Phi(h_0)$ changes sign at $t_1 = \nu\gamma^2/(4\theta)$ being positive for $t < t_1$ and negative for $t > t_1$. The second expression multiplying the pdf $\Phi'(h_0)$ changes sign at $t_2 = -\gamma x_0/2$ only if γ and $x_0 = VaR_\alpha$ are such that $\gamma x_0 < 0$ (note, we are interested in partitioning the quadratures for $t \in [0, t_0(x_0)]$, that is for positive t's, on two intervals which union covers the quadratures interval).

If $\gamma x_0 > 0$ then the second expression in $u'(t)$ does not change sign. Hence, the derivative has a unique zero, respectively the integrand function in the quadratures has a unique maximum, for t "near" t_1, that is, for $t < t_1$ because the second expression in $u'(t)$ takes on negative values which shift the t_1 zero to the left. We note that in both cases for θ, i.e. for $(\nu + 2)/2$ and $\nu/2$, the t_1 value is close to the $\gamma^2/2$ approximation which we suggest in this proposition, and which is numerically tested.

If $\gamma x_0 < 0$ then the derivative $u'(t)$ can change sign (eventually more than once) for t between $t_1 \approx \gamma^2/2$ and $t_2 = -\gamma x_0/2$. Otherwise, for $t < \min(t_1, t_2)$ the derivative $u'(t)$ takes on positive values, and for $t > \max(t_1, t_2)$ the derivative has negative values. Therefore there is at least one maximum for the integrand function $u(t)$ between t_1 and t_2, that is, for t belonging to the interval $[\min(t_1, t_2), \max(t_1, t_2)]$.

We combine the conclusions from the above two cases, and we approximate the location of the maximum with $t_1 \approx \gamma^2/2$ for all cases.

□

Based on the proposition and the numerical experiments, we conclude that every case of numerical integration in the formulas for VaR and AVaR must be partitioned in two quadratures at $\gamma^2/2$. This is especially important for near symmetric skewed-T distributions, that is when γ goes to zero. This completely resolves the numerical issues arising in the analytical VaR calculation. However, in the analytical AVaR calculation we have to deal with the Bessel function evaluation involved in our formula.

We use the following two asymptotic properties of the modified Bessel function of the third kind

$$K_\lambda(x) \longrightarrow \sqrt{\frac{\pi}{2x}} e^{-x}$$

for large $x >> |\lambda^2 - 1/4|$, and

$$K_\lambda(x) \to \frac{\Gamma(\lambda)}{2} \left(\frac{2}{x} \right)^\lambda$$

for small positive $x << \sqrt{\lambda + 1}$. The first asymptotic is especially important for large γ. For such γ's we have that β goes to plus infinity and β has the order of z_0. In this case we see that the Beseel part

$$K_{\lambda-1}(\beta) \left(\frac{2}{\beta} \right)^{\lambda-1} e^{\gamma x_0}$$

in the KI expression tends to zero which significantly helps in the numerical calculations because, otherwise, one might have to deal with an undefined numeric expression looking like zero times infinity. On the other side when evaluating the above expression for small γ then the second asymptotic significantly improves the accuracy in the analytical AVaR calculation.

4 Conclusions

We develop analytical formulas for computing the α level VaR and AVaR for a random variable X having the asymmetric Student-t distribution, also known as the skewed-T distribution. It is defined as a normal mixture with inverse Gamma distribution. The distribution pdf and the AVaR formula require the Bessel function calculation.

The analytical formulas are tested and they appear to be accurate for different confidence levels α. The parameters of the skewed-T distribution $X \sim t_d(\nu, \mu, \sigma, \gamma)$ are also varied. For the normalized random variable $(X-\mu)/\sigma$ it is important to vary the degrees of freedom parameter ν and the ratio γ/σ when the achieved numerical accuracy is tested against very large sample estimates. We tested the analytical formulas for ν in the range $[4, 400]$ where the results for $\nu > 300$ are approximated very well with $\nu = 300$. The test range for the ratio γ/σ is $[10^{-4}, 10^2]$. In all tested cases the analytical formulas yield results which differ from ten million sample size estimates by less than one percent.

The derived analytical formulas are very useful if only a small number of sample observations is available. In this case we find that the sample estimates tend to underestimate the heavy tail extreme quantiles. On the other side, for large sample size we find that the sample estimates tend to overestimate the light (short) tail extreme quantiles. Further qualitative and quantitative analysis could be performed on the derived formulas in a future research concerning their applications in modeling financial time series.

Appendix

The VaR formula for $\gamma > 0$ (Proof of Theorem 1)

The result for the α level Value-at-Risk, $x_0 = VaR_\alpha$ of a skewed-T random variable, derived in the main text, says that x_0 is the unique zero of the following equation

$$g(x_0) = -\alpha + \frac{2C\sqrt{\pi}}{\gamma}\int_0^\infty t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}}\,\Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right) dt = 0, \text{ if } \gamma > 0,$$

where the zero, x_0 of g, is sought in the interval $[-9/\gamma, 0]$ provided $\alpha < \alpha_0$, or in the interval $[0, +\infty]$ provided $\alpha > \alpha_0$. Furthermore, $g(\cdot)$ is an increasing function of x_0 which actually implies the uniqueness of the zero.

Here we prove this fact. We begin with the VaR definition

$$1 - \alpha = \int_{x_0}^\infty f(x)dx$$

which is rewritten in

$$g(x_0) = 1 - \alpha - \int_{x_0}^{\infty} f(x)dx.$$

Next, we simplify the integral

$$\begin{aligned}\int_{x_0}^{\infty} f(x)dx &= \int_{x_0}^{\infty} C \int_0^{\infty} t^{-\lambda-1} e^{-t-\frac{\nu\gamma^2}{4t}} e^{\gamma x - \frac{(\gamma x)^2}{4t}} dt dx \\ &= \frac{C}{\gamma} \int_0^{\infty} t^{-\lambda-1} e^{-t-\frac{\nu\gamma^2}{4t}} \int_{x_0}^{\infty} e^{\gamma x - \frac{(\gamma x)^2}{4t}} d(\gamma x) dt.\end{aligned}$$

The change of variables $z = \gamma x$, $z_0 = \gamma x_0$, in the inner integral yields

$$\begin{aligned}\int_{x_0}^{\infty} f(x)dx &= \frac{C}{\gamma} \int_0^{\infty} t^{-\lambda-1} e^{-t-\frac{\nu\gamma^2}{4t}} \int_{z_0}^{\infty} e^{z-\frac{z^2}{4t}} dz dt \\ &= \frac{2C\sqrt{\pi}}{\gamma} \int_0^{\infty} t^{-\lambda-1/2} e^{-\frac{\nu\gamma^2}{4t}} \left[1 - \Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right)\right] dt.\end{aligned}$$

We now use the following identity

$$\frac{2C\sqrt{\pi}}{\gamma} \int_0^{\infty} t^{-\lambda-1/2} e^{-\frac{\nu\gamma^2}{4t}} dt = 1$$

(note, the above right-hand side, 1, must be replaced with -1 if $\gamma < 0$). We substitute back $\lambda = (\nu+1)/2$, and we obtain

$$\int_{x_0}^{\infty} f(x)dx = 1 - \frac{2C\sqrt{\pi}}{\gamma} \int_0^{\infty} t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right) dt$$

which completes the proof.

The approximation VaR formula for $\gamma > 0$ (Proof of Corollary 1)

We note that for t such that

$$\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t} < -6.$$

the tail of the last integral (in the above proof) becomes infinitely small. This is so because the standard normal cdf satisfies $\Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right) < \Phi(-6) < 10^{-9}$, and the integral from the remaining integrand function is bounded by one over the tail of the integral (this statement is rigorously proved below). The tail is on the interval $t \in [t_0(x_0), +\infty]$ where $t_0(x_0)$ satisfies the above inequality. The last inequality is equivalent to a quadratic inequality with respect to $\sqrt{t}$

$$2t - 6\sqrt{2t} - \gamma x_0 > 0$$

which is true for

$$t > t_0(x_0) \equiv \left(\frac{3\sqrt{2} + \sqrt{18 + 2\gamma x_0}}{2}\right)^2$$

(which coincides with the notation $t_0(x)$ in the main text). We partition the last integral on $[0,\infty]$ as integral on $[0, t_0(x_0)]$ plus integral on $[t_0(x_0), \infty]$. For the latter we argue above that it is infinitely small (basically, it is the error term $R(t_0(x_0))$). This is so because we have the following inequality

$$\begin{aligned} R(t_0(x_0)) &\equiv \frac{2C\sqrt{\pi}}{\gamma} \int_{t_0(x_0)}^{\infty} t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right) dt \\ &\leq \frac{2C\sqrt{\pi}}{\gamma} \Phi(-6) \int_{t_0(x_0)}^{\infty} t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} dt \end{aligned}$$

The last expression simplifies to, and is bounded by $\Phi(-6) < 10^{-9}$

$$\Phi(-6) \frac{\gamma\left(\frac{\nu\gamma^2}{4t_0(x_0)}; \frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \leq \Phi(-6)$$

where $\gamma\left(\frac{\nu\gamma^2}{4t_0(x_0)}; \frac{\nu}{2}\right)$ is the lower incomplete Gamma function. Therefore

$$\int_{x_0}^{\infty} f(x)dx \approx 1 - \frac{2C\sqrt{\pi}}{\gamma} \int_0^{t_0(x_0)} t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right) dt$$

which completes the proof of the approximation formula for $g(x_0)$. Rigorously we have

$$g(x_0) = -\alpha + \frac{2C\sqrt{\pi}}{\gamma} \int_0^{t_0(x_0)} t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right) dt + R(t_0(x_0))$$

and,

$$g(x_0) \geq -\alpha + \frac{2C\sqrt{\pi}}{\gamma} \int_0^{t_0(x_0)} t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right) dt$$

where we chose the lower bound of $g(x_0)$ as its approximation.

Furthermore, (recall, we are in the case $\gamma > 0$), we note that if $\alpha > \alpha_0$, i.e., $x_0 > 0$, then the quadratic inequality is true for $t > t_0(x_0)$. And, if $\alpha < \alpha_0$, i.e., $x_0 < 0$, then the quadratic inequality is true for $t > t_0(0)$ provided $18 + 2\gamma x_0 > 0$. Otherwise, if $18 + 2\gamma x_0 < 0$ then the quadratic inequality is true for any t. Hence, the search for negative x_0 should be performed only for $x_0 > -9/\gamma$. Finally, the approximation for $g(x_0)$ increases in x_0 because from the expression we have for $g(x_0)$ we see that $\Phi(\cdot)$ increases in x_0 and, the integral in the $g(x_0)$ expression is an integral from nonnegative function on the interval $[0, t_0(x_0)]$ where the right end $t_0(x_0)$ of the interval also increases with respect to x_0.

The $AVaR_1$ formula for skewed-T (Proof of Theorem 2)

The proof for the following formula

$$AVaR_1 = \frac{\gamma\nu}{(1-\alpha)(\nu-2)} + \frac{4C}{(1-\alpha)\gamma^2} KI$$

is presented below.

$$
\begin{aligned}
AVaR_1 &= E[X|X > x_0] \\
&= \frac{1}{P(X > x_0)} \int_{x_0}^{\infty} x f(x) dx \\
&= \frac{1}{(1-\alpha)} \int_{x_0}^{\infty} x \frac{\nu^{\lambda}\gamma^{2\lambda}}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}2^{2\lambda}} \int_0^{\infty} t^{-\lambda-1} e^{-t-\frac{\nu\gamma^2}{4t}} e^{\gamma x - \frac{(\gamma x)^2}{4t}} dt dx \\
&= \frac{C}{(1-\alpha)\gamma^2} \int_0^{\infty} t^{-\lambda-1} e^{-t-\frac{\nu\gamma^2}{4t}} \int_{x_0}^{\infty} (\gamma x) e^{\gamma x - \frac{(\gamma x)^2}{4t}} d(\gamma x) dt
\end{aligned}
$$

We set $z = \gamma x$, $z_0 = \gamma x_0$, and simplify the inner integral, first, with integration by parts

$$
\begin{aligned}
\int_{z_0}^{\infty} z e^{z - \frac{z^2}{4t}} dz &= -2t \int_{z_0}^{\infty} e^z d\left(e^{\frac{-z^2}{4t}}\right) \\
&= 2t\left(e^{z_0 - \frac{z_0^2}{4t}} + \int_{z_0}^{\infty} e^{z - \frac{z^2}{4t}} dz\right)
\end{aligned}
$$

and, second, with change of variables technique $h = \frac{z}{\sqrt{2t}} - \sqrt{2t}$, $h_0 = \frac{z_0}{\sqrt{2t}} - \sqrt{2t}$ leading to closed-form result with standard normal cdf

$$
\begin{aligned}
\int_{z_0}^{\infty} z e^{z - \frac{z^2}{4t}} dz &= 2t\left(e^{z_0 - \frac{z_0^2}{4t}} + e^t\sqrt{2t}\int_{h_0}^{\infty} e^{-\frac{h^2}{2}} dh\right) \\
&= 2t\left(e^{z_0 - \frac{z_0^2}{4t}} + e^t\sqrt{2t}\sqrt{2\pi}(1 - \Phi(h_0))\right)
\end{aligned}
$$

Hence, the conditional expectation of the right tail of the skewed-T distribution becomes

$$
\begin{aligned}
AVaR_1 &= \frac{2C}{(1-\alpha)\gamma^2} \int_0^{\infty} t^{-\lambda} e^{-t-\frac{\nu\gamma^2}{4t}} \left(e^{z_0 - \frac{z_0^2}{4t}} + e^t\sqrt{2t}\sqrt{2\pi}(1 - \Phi(h_0))\right) dt \\
&= \frac{2C}{(1-\alpha)\gamma^2}(E_1 + E_2 - E_3),
\end{aligned}
$$

where

$$
\begin{aligned}
E_1 &= e^{z_0} \int_0^{\infty} t^{-\lambda} e^{-t-\frac{\nu\gamma^2}{4t}-\frac{z_0^2}{4t}} dt \\
E_2 &= 2\sqrt{\pi} \int_0^{\infty} t^{-\lambda+1/2} e^{-\frac{\nu\gamma^2}{4t}} dt \\
E_3 &= 2\sqrt{\pi} \int_0^{\infty} t^{-\lambda+1/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi(h_0) dt
\end{aligned}
$$

We combine the integral representation of the Bessel function with one earlier notation $\beta = \sqrt{\nu\gamma^2 + (\gamma x_0)^2}$ (recall $z_0 = \gamma x_0$) which simplifies E_1 to a closed-form result

$$
\begin{aligned}
E_1 &= e^{z_0}\int_0^\infty t^{-\lambda}e^{-t-\frac{\beta^2}{4t}}dt \\
&= 2e^{z_0}\left(\frac{2}{\beta}\right)^{\lambda-1}K_{\lambda-1}(\beta)
\end{aligned}
$$

We note that the result for E_1 (along with the $\frac{2C}{(1-\alpha)\gamma^2}$ multiplier) corresponds to the Bessel function term in KI

$$
\frac{4C}{(1-\alpha)\gamma^2}K_{\lambda-1}(\beta)\left(\frac{2}{\beta}\right)^{\lambda-1}e^{\gamma x_0}
$$

in the formula for $AVaR_1$.

Next, we show that E_2 simplifies to the first term in the $AVaR_1$ formula. We apply the Gamma function definition $\Gamma(u)=\int_0^\infty v^{u-1}e^{-v}dv$ for argument $u=\lambda-3/2$ with change of variables $v=\frac{\nu\gamma^2}{4t}$. Hence

$$
\begin{aligned}
E_2 &= 2\sqrt{\pi}\int_0^\infty t^{-\lambda+1/2}e^{-\frac{\nu\gamma^2}{4t}}dt \\
&= 2\sqrt{\pi}\left(\frac{4}{\nu\gamma^2}\right)^{\lambda-3/2}\Gamma(\lambda-3/2) \\
&= \sqrt{\pi}\,\frac{2^{2\lambda-2}\Gamma(\lambda-1/2)}{\nu^{(\lambda-3/2)}\gamma^{2\lambda-3}(\lambda-3/2)} \\
&= \frac{\gamma^3\nu}{2(\nu-2)C}
\end{aligned}
$$

where the second to the last equality comes from the following property $\Gamma(u)=(u-1)\Gamma(u-1)$. For the last equality we use the definition of the notation C and recall that $\lambda=(\nu+1)/2$. We note that adjusting the last result for E_2 with the multiplier $\frac{2C}{(1-\alpha)\gamma^2}$ yields the first term in the $AVaR_1$ formula.

Finally, we deal with the E_3 expression

$$
\frac{E_3}{2\sqrt{\pi}}=\int_0^\infty t^{-\lambda+1/2}e^{-\frac{\nu\gamma^2}{4t}}\Phi(h_0)dt
$$

which is an integral on infinite interval. We note that the result for E_3 (along with the $\frac{2C}{(1-\alpha)\gamma^2}$ multiplier) corresponds to the third term in the $AVaR_1$ formula, that is, to the integral expression in KI. This completes the proof of the theorem.

The approximation formula for $AVaR_1$ (Proof of Corollary 3)

We keep work on the E_3 expression from the end of the previous proof. We will approximate the integral with one on a finite interval. We note that the argument, h_0, of the standard normal cdf tends to minus infinity as t goes to plus infinity (recall $h_0(t)=\frac{z_0}{\sqrt{2t}}-\sqrt{2t}$). Hence, for sufficiently large t we have

$\Phi(h_0)$ going to zero. Technically, sufficiently large t can be determined (as in the proof of Theorem 1) from $\Phi(-6) < 10^{-9}$. The inequality $h_0 < -6$ is true for $t > t_0(x_0)$ where the last notation $t_0(x_0)$ is already specified in the main text and in the proof of Theorem 1. We so obtain

$$\frac{E_3}{2\sqrt{\pi}} = \int_0^{t_0(x_0)} t^{-\lambda+1/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi(h_0) dt + \int_{t_0(x_0)}^{\infty} t^{-\lambda+1/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi(h_0) dt$$

The first integral corresponds to the third term in the $AVaR_1$ formula, that is, to the integral expression in KI (*"here we complete the proof of this formula"*). The second integral "vanishes" because it simplifies similarly to the proof of Theorem 1. We have that $h_0(t) \leq h_0(t_0(x_0)) = -6$ for $t \geq t_0(x_0)$, hence the integral is bounded by

$$\int_{t_0(x_0)}^{\infty} t^{-\lambda+1/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi(h_0) dt \leq \Phi(-6) * \int_{t_0(x_0)}^{\infty} t^{-\lambda+1/2} e^{-\frac{\nu\gamma^2}{4t}} dt$$

As it is already done earlier in this proof, the last expression must be adjusted by the multipliers $\frac{2C}{(1-\alpha)\gamma^2}$ and $2\sqrt{\pi}$. Hence, after simplification (similar to the one in the proof of Theorem 1) we obtain that the second integral is bounded by

$$\Phi(-6) * \frac{\nu\gamma}{(1-\alpha)(\nu-2)} * \frac{\gamma\left(\frac{\nu\gamma^2}{4t_0(x_0)}; \frac{\nu}{2} - 1\right)}{\Gamma\left(\frac{\nu}{2} - 1\right)}$$

where the last multiplier is bounded by 1, the second multiplier is a constant for given distribution parameters and AVaR level α, and the first multiplier is bounded by 10^{-9}. Therefore, we assume that the error made in the transition from integral on infinite interval to integral on finite interval is infinitely small.

The "big" M formula (Proof of Proposition 1)

The proof for the following formula is presented below

$$M = \frac{2d + 3\sqrt{8d}}{(-\gamma)}$$

where

$$d = \frac{\nu\gamma^2}{4} \left[\epsilon\Gamma\left(\frac{\nu+2}{2}\right)\right]^{-2/\nu}$$

in the case $\alpha < \alpha_0$ and $\gamma < 0$.

In this case we look for the unique zero x_0 of

$$g(x_0) = 1 - \alpha + \frac{2C\sqrt{\pi}}{\gamma} \int_0^{t_0(x_0)} t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right) dt = 0,$$

in $(-\infty, 0]$, and we prove that for a given small positive number ϵ such that $\alpha > \epsilon$ the zero search can be performed on $[-M, 0]$ rather than on $(-\infty, 0]$. We note that we are in the very left end of the heavy left tail of the skewed-T distribution, that is, the small ϵ represent the probability for being in $(-\infty, -M]$. In the main text we said that $g(\cdot)$ is an increasing function from $g(-\infty) = -\alpha < 0$ to $g(0) = -\alpha + \alpha_0 > 0$. So, we now prove that

$$g(-M) \leq -\alpha + \epsilon,$$

that is, $g(-M) < 0$ provided that ϵ is chosen such that $(\epsilon < \alpha)$ it is less than the user specified confidence level α for VaR_α.

We utilize the notations

$$F(t,k) = t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \frac{e^{-k^2/2}}{\sqrt{2\pi}}$$
$$h_0(t) = \frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}$$

and the definition of the standard normal cdf $\Phi(\cdot)$ to rewrite the integral in the $g(x_0)$ expression as follows

$$\int_0^{t_0(x_0)} \int_{-\infty}^{h_0(t)} F(t,k) dk dt.$$

Next, we change the order of integration, and present the integral as a sum of two integrals

$$\int_{-\infty}^{-6} \int_0^{t_0(x_0)} F(t,k) dt dk + \int_{-6}^{\infty} \int_0^{h_0^{-1}(k)} F(t,k) dt dk$$

where $t = h_0^{-1}(k)$ is the inverse function of $k = h_0(t)$. The two functions are defined on the intervals $k \in [-6, +\infty)$ and $t \in [0, t_0(x_0)]$ respectively (note, the inverse function exists because $h_0(\cdot)$ is a monotone function). The first double integral is bounded by

$$\frac{\Gamma\left(\frac{\nu\gamma^2}{4t_0(x_0)}; \frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \Phi(-6)$$

which is a product of (positive) multiplier less than one (i.e. the ratio of upper incomplete Gamma function and the Gamma function) and, technically, the zero value $\Phi(-6) \approx 10^{-9}$. We so focus only on the second double integral in the last expression for $g(x_0)$. Hence,

$$\begin{aligned} g(x_0) &= 1 - \alpha - \frac{1}{\Gamma\left(\frac{\nu}{2}\right)} \int_{-6}^{\infty} \left[\Gamma\left(\frac{\nu}{2}\right) - \gamma\left(\frac{\nu\gamma^2}{4h_0^{-1}(k)}; \frac{\nu}{2}\right)\right] \frac{e^{-k^2/2}}{\sqrt{2\pi}} dk \\ &= -\alpha + \frac{1}{\Gamma\left(\frac{\nu}{2}\right)} \int_{-6}^{\infty} \gamma\left(\frac{\nu\gamma^2}{4h_0^{-1}(k)}; \frac{\nu}{2}\right) \frac{e^{-k^2/2}}{\sqrt{2\pi}} dk \end{aligned}$$

where $\gamma\left(\cdot; \frac{\nu}{2}\right)$ is the lower incomplete Gamma function, i.e., $\Gamma\left(\frac{\nu}{2}\right) = \gamma\left(\cdot; \frac{\nu}{2}\right) + \Gamma\left(\cdot; \frac{\nu}{2}\right)$. Note, for the pair of double integrals in $g(x_0)$ we utilized the identity

$$\frac{2C\sqrt{\pi}}{\gamma} \left(\frac{\nu\gamma^2}{4}\right)^{-\nu/2} \Gamma\left(\frac{\nu}{2}\right) = -1$$

for $\gamma < 0$ (otherwise, the above expression simplifies to plus one for $\gamma > 0$). Next we have

$$g(x_0) = -\alpha + \frac{1}{\Gamma\left(\frac{\nu}{2}\right)} \int_{-6}^{6} \gamma\left(\cdot; \frac{\nu}{2}\right) \frac{e^{-k^2/2}}{\sqrt{2\pi}} dk + \int_{6}^{\infty} \frac{\gamma\left(\cdot; \frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{e^{-k^2/2}}{\sqrt{2\pi}} dk$$

where the second integral is technically equal zero for reasons described earlier with respect to the first double integral in $g(x_0)$. Therefore, we focus on bounding the integral on finite interval $[-6, 6]$. We note that the lower incomplete gamma function in this integral, $\gamma\left(\cdot; \frac{\nu}{2}\right)$, is an increasing function of k in its argument $\frac{\nu\gamma^2}{4h_0^{-1}(k)}$. Hence,

$$\begin{aligned} g(x_0) &\leq -\alpha + \frac{1}{\Gamma\left(\frac{\nu}{2}\right)} \int_{-6}^{6} \gamma\left(\cdot; \frac{\nu}{2}\right)\Big|_{k=6} \frac{e^{-k^2/2}}{\sqrt{2\pi}} dk \\ &= -\alpha + \frac{\gamma\left(\frac{\nu\gamma^2}{4h_0^{-1}(6)}; \frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \end{aligned}$$

Next we bound the lower incomplete gamma function $\gamma(u; a)$ by

$$\gamma(u; a) = \int_0^u t^{a-1} e^{-t} dt \leq u^a / a.$$

Hence,

$$g(x_0) \leq -\alpha + \frac{\left(\frac{\nu\gamma^2}{4h_0^{-1}(6)}\right)^{\nu/2}}{\frac{\nu}{2}\Gamma\left(\frac{\nu}{2}\right)} \equiv l(x_0)$$

where

$$\begin{aligned} h_0^{-1}(6) &= \left.\frac{(\sqrt{k^2 + 4\gamma x_0} - k)^2}{8}\right|_{k=6} \\ &= \frac{(\sqrt{9 + \gamma x_0} - 3)^2}{2} \end{aligned}$$

Here we may observe that

$$h_0^{-1}(6)\big|_{x_0=-M} = d$$

where d is specified in the M definition.

Finally, some tedious algebraic manipulations show that $l(-M) = -\alpha + \epsilon$ which completes the proof for the "big" M formula.

Remark: If we decide that we do not have to ignore (we do not want to ignore) the two infinite small terms (integrals) in the proof (involving a constant times $\Phi(-6)$ where the absolute value of that constant is less than one) then we may choose ϵ such that $\epsilon < \alpha - 2 * 10^{-9}$ for some specified in advance VaR level α.

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