Module S-5/1 - Part 1 -
Asset Liability Management

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Literature Recommendations

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2) Stavros A. Zenios, Lecture Notes: Mathematical modeling and its application in finance, http://www.hermes.ucy.ac.cy/zenios/teaching/399.001/index.html

3) F. Fabozzi and A. Konisbi, The Handbook of Asset/Liability Management: State-of-Art Investment Strategies, Risk Controls and Regulatory Required, Wiley.

4) Svetlozar Rachev and Stefan Mittnik, Stable Paretian Models in Finance, John Wiley & Sons Ltd., 2000

Overview

1) **Introduction**

2) **Optimization and Risk**

3) **Optimization Problems, Stochastic Programing and Scenario Analysis**

4) Modeling of the Risk Factors

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1) Introduction

1.1) Asset and Liability Streams

1.2) Examples

1.3) Major Tasks of ALM

Introduction

Asset and Liability Streams

Asset liability management (ALM) attempts to find the optimal investment strategy under uncertainty in both:

- the asset streams
- the liability streams.

$\Rightarrow$ The simultaneous consideration of assets and liabilities can be advantageous when they have common risk factors.

Introduction

Asset and Liability Streams

- Traditionally, banks and insurance companies used accrual accounting for essentially all their assets and liabilities.
- On the one hand they would take on liabilities, such as deposits, life insurance policies or annuities.
- On the other hand they would invest the proceeds from these liabilities in assets such as loans, bonds or real estate.
- All assets and liabilities were held at book value.
- Doing so the approach **disguised possible risks** arising from how the assets and liabilities were structured.

Introduction

Example:

- A bank borrows USD 100 Mio at 3.00% for a year and lends the same money at 3.20% to a highly-rated borrower for 5 years.
- For simplicity, we assume that all interest rates are annually compounded and all interest accumulates to the maturity of the respective obligations.
- The net transaction appears profitable, since the bank is earning a 20 basis point spread.
- However, the transaction also entails considerable risk:
- At the end of a year, the bank will have to find new financing for the loan, which will have 4 more years before it matures. If interest rates have risen, the bank may have to pay a higher rate of interest on the new financing than the fixed 3.20 it is earning on its loan.

Introduction

Example:

Suppose, for example that at the end of a year, an applicable 4-year interest rate is 6.00%

Accrual accounting does not recognize the problem. The book value of the loan (the bank's asset) is:

$$100 Mio \cdot 1.032 = 103.2 Mio$$

The book value of the financing (the bank's liability) is:

$$100 Mio \cdot 1.030 = 103.0 Mio$$

Based upon accrual accounting, the bank earned USD 200,000 in the first year.

Introduction

Example:

However, market value accounting recognizes the bank's predicament. The respective market values of the bank's asset and liability are:

$$\frac{100Mio \cdot 1.032^5}{1.060^4} = 92.72Mio$$

Hence, from a market-value accounting standpoint, the bank has lost USD 10.28 Mio.

- The bank is in trouble, and the market-value loss reflects this.
- Ultimately, accrual accounting will recognize a similar loss when the bank will have to secure financing for the loan at the new higher rate.

Introduction

Example:

The problem in this example was caused by a **mismatch between assets and liabilities**.

We conclude that it is necessary to find the optimal investment strategy under consideration of the time horizons of both

- the asset side
- the liability side

Introduction

Example - Changing Economic Conditions

- Until the end of the 1970s interest rates in developed countries experienced only modest fluctuations, so losses due to asset-liability mismatches were small or trivial.
- Because yield curves were generally upward sloping, banks could earn a spread by borrowing short and lending long.
- Things started to change in the 1970s, which ushered in a period of volatile interest rates that continued into the early 1980s.
- Managers of many firms, who were accustomed to thinking in terms of accrual accounting, were slow to recognize the emerging risk.
- Because the firms used accrual accounting, the result was not so much bankruptcies as crippled balance sheets. Firms gradually accrued the losses over the subsequent 5 or 10 years.

Introduction

Example - Changing Economic Conditions

- One of the victims of the changing conditions is the US mutual life insurance company the Equitable.
- During the early 1980s, the USD yield curve was inverted, with short-term interest rates spiking into the high teens.
- The Equitable sold a number of long-term guaranteed interest contracts (GICs) guaranteeing rates of around 16% for periods up to 10 years.
- Equitable invested the assets short-term to earn the high interest rates guaranteed on the contracts.
- Short-term interest rates soon came down. When the Equitable had to reinvest, it couldn't get nearly the interest rates it was paying on the GICs.
- Eventually Equitable was acquired by the Axa Group.

Introduction

Example: Fixed-Income Securities

Classic ALM frameworks for constructing portfolios of fixed-income securities: **dedication** and **immunization**:

- Dedication: assumes that the future liability payments are deterministic and finds an allocation such that bond income is sufficient to cover the liability payments in each time period.
- Immunization: the portfolio is constructed by matching the present values and interest rate sensitivities of the assets and liabilities. Allocation that hedges against a small parallel shift in the term structure of interest rates.

$\Rightarrow$ Neglection of the stochastic nature of interest rates and liabilities and the dynamic nature of investing.

Introduction

Major Tasks of ALM

Find Adequate Measures and Models for Risk Quantification

- Adequate Risk Measures like
  - Value-at-Risk
  - Expected Shortfall
  - Alternative Measures
- Properties of Risk Measures

Introduction

Major Tasks of ALM

Capture the behavior of returns by an adequate distribution and model:

- Finding an adequate distribution
    - Are asset returns normally distributed?
    - Alternative heavy-tailed distributions
- Time-Series Models
    - autocorrelation in the returns
    - conditional variance etc.

Introduction

Major Tasks of ALM

Capture the dependence structure between the risk factors:

- Fit Multivariate Distributions
- Dependence Modeling for univariate distributions
  - Correlation as measure of dependence
  - Copulas

Introduction

Major Tasks of ALM

Capture the dynamic and stochastic characteristics of assets and liabilities:

- Scenario analysis techniques
- Stochastic control methods
- Stochastic programming

1) Optimization and Risk

1.1) Risk Measures

1.2) Properties of Risk Measures

1.3) Risk-Return Optimization

Optimization and Risk

Risk-Return Optimization

The goal of risk-return optimization is to **optimize a tradeoff** between the risk and return.

Major Issues:

- Adequate Risk Measures
- Portfolio Optimization Problems
- Techniques for Optimization

Risk Measures

Standard measure of Risk

The standard measure of risk for a portfolio of equities suggested by Markowitz is the **variance** of the return:

Portfolio consists of

- n assets with corresponding risky returns $R = (r_1, ..., r_n)'$.
- portfolio weights $\omega = (\omega_1, ..., \omega_n)'$ such that $\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$.

$\Rightarrow$ The risk associated with the portfolio return $r_p = \omega' R$ is given by

$$\sigma_p^2 = \omega' \Sigma \omega$$

where Σ is the covariance matrix of R.

Risk Measures

Standard measure of Risk - Modifications

Criticism of the variance as standard risk measure:

- Variance **penalizes both large gains and large losses**. However, only large losses are critical for investors or the survival of the institution.
- Financial returns are typically **heavy-tailed**, and in that case, the variance does not even exists.

Risk Measures

Standard measure of Risk - Modifications

Possible Modification: asymmetric risk measure that accounts only for large losses is the **semivariance**:

$$\mathrm{E}\left([\omega' \mathrm{E}(R) - \omega' R]^{+} \right)^{2}.$$

Problem: numerical optimization of the semivariance is difficult.

Alternative modification: use **downside formula** which measures the degree that the returns are distributed below some target return r^*:

$$\mathrm{E}\left([r^* - \omega' R]^{+} \right)^{2}.$$

Risk Measures

Standard measure of Risk - Modifications

Use the *mean absolute deviation* (MAD) of the portfolio as risk measure:

$$m_p = \mathrm{E}\left|\omega' R - \omega' \mathrm{E}(R)\right|,$$

Use the **scale parameter** of a heavy-tailed distribution as measure of risk (see later sections).

Risk Measures

Tail Measures

Other risk measures rely only on the **tail of the distribution**: modeling of the probability of extreme events becomes more important.

Most prominent measures:

- **Value at Risk (VaR)**
- **Conditional Value at Risk (CVaR)**

Risk Measures

Value at Risk

For a given confidence level $\beta \in (0,1)$, VaR is the minimum value of the loss, or negative return, that is exceeded no more than 100(1-β)% of the time:

$x \in \mathbb{R}^n$: given decision on asset allocation $L(x) \in \mathbb{R}$: random variable representing loss, or negative return, for each x
$\Psi_L(x,\zeta)$: the distribution function for $L(x)$: $\Psi_L(x,\zeta) = \mathrm{P}\left(L(x) \le \zeta\right)$

For a given decision x, the Value at Risk at confidence level β is given by:

$$\mathrm{VaR}_\beta(x) = \inf\left\{\zeta \middle| \Psi_L(x,\zeta) \ge \beta\right\}.$$

Risk Measures

Conditional Value at Risk

Define a random variable $T_\beta(x)$ on the β-tail of the loss $L(x)$ through the distribution function:

$$\Psi_{T_\beta}(x,\zeta) = \begin{cases} 0 & \zeta < \mathrm{VaR}_\beta(x) \\ \frac{\Psi_L(x,\zeta)-\beta}{1-\beta} & \zeta \geq \mathrm{VaR}_\beta(x) \end{cases} \tag{1}$$

The *Conditional Value at Risk* at confidence level β is the mean of the tail random variable $T_\beta(x)$ with distribution function (1):

$$\mathrm{CVaR}_\beta(x) = \mathrm{E}\left(T_\beta(x)\right).$$

Risk Measures

VaR and CVaR

CVaR is closely related to the conditional expectation beyond VaR:

$$\mathrm{E}\left(L(x) | L(x) \geq \mathrm{VaR}_\beta(x)\right) \leq \mathrm{CVaR}_\beta(x) \leq \mathrm{E}\left(L(x) | L(x) > \mathrm{VaR}_\beta(x)\right). \quad (2)$$

If there is no discontinuity in the distribution function of $L(x)$ at $\mathrm{VaR}_\beta(x)$, then equality holds in equation (2).
For this reason, CVaR is also sometimes called the **Expected Tail Loss** (ETL).

Risk Measures

Alternative Risk Measures

The literature suggests several alternative risk measures:

- RAROC
- Rachev's Ratio
- ....

Further examples will be discussed in the lecture.

Properties of Risk Measures

Coherent Risk Measures

To help define a sensible risk measure, properties that are required of a *coherent* risk measure are introduced (see e.g. Artzner et al):

i. sub-additive: $\rho(v + v') \leq \rho(v) + \rho(v')$,

ii. positive homogeneous: $\rho(\lambda v) = \lambda \rho(v), \quad \forall \lambda \geq 0$,

iii. translation invariant: $\rho(v + c) = \rho(v) + c, \quad \forall c \in \mathbb{R}$, and

iv. monotonous: $\rho(v) \geq 0, \quad \forall v \geq 0$.

Hereby is:
V the space of real-valued random variables, $\rho : V \longrightarrow \mathbb{R}$ risk measure,
$v, v' \in V$ random variables are thought of as losses

Coherence of VaR

VaR does not satisfy these properties in general because VaR is not sub-additive:

Example:

$\Rightarrow$ Lack of subadditivity is very undesirable because diversification is not promoted.

Properties of Risk Measures

Coherence of CVaR

The coherence of the set of random variables $\{L(x)\}$ can be stated as a function of x when $L(x)$ is linear:

$$L(x) = x_1 Y_1 + \ldots + x_n Y_n.$$

In this situation, Y_i might be a random variable representing an individual asset loss, and $L(x)$ is a random variable representing the total portfolio loss. Coherence of $\text{CVaR}_\beta(x)$ in this framework means

1. $\text{CVaR}_\beta(x)$ is sublinear in x,
2. $\text{CVaR}_\beta(x) = c$ when $L(x) = c \in \mathbb{R}$, and
3. $\text{CVaR}_\beta(x) \leq \text{CVaR}_\beta(x')$ when $L(x) \leq L(x')$.

Remarks on Coherent Risk Measures

- Sub-additivity and positive homogeneity guarantee that a coherent risk measure is convex.
- A lack of convexity of VaR contributes to numerical difficulties in optimization.
- For the special class of elliptical distribution, VaR is sub-additive and coherent.
- VaR is easy to work with when normality of distributions is assumed, but financial data is typically heavy-tailed.
- In addition to coherence, CVaR has a representation that is practical in minimization problems with scenarios generated from any distributional assumption.

Risk-Return Optimization

The mean-variance optimization problem

The classical mean-variance optimization problem is to minimize the risk of the portfolio for a minimum level of expected return:

$$\begin{aligned} \min_{\omega} \quad & \omega' \Sigma \omega \\ \text{s.t.} \quad & \omega' \mu = \mu_0, \\ & \textstyle\sum_{i=1}^{n} \omega_i = 1. \end{aligned} \tag{3}$$

Risk-Return Optimization

The mean-variance optimization problem

If the risky returns R are assumed to follow a multivariate normal distribution $N(\mu, \Sigma)$, the portfolio return $r_p = \omega' R$ is also normally distributed with:

- mean $\mu_p = \omega' \mu$
- variance $\sigma_p^2 = \omega' \Sigma \omega$.

The solution to the above problem is easily solved with Lagrangian techniques.

Example:

Risk-Return Optimization

The mean-variance optimization problem

A drawback of optimization problem (3) is that it requires a large number of parameters to be estimated:

- Assume n risky assets: the covariance matrix consists of $n(n+1)/2$ elements.
- For example: if the universe of assets consists of the S&P500, over 125,000 variances/covariances must be estimated.

Risk-Return Optimization

The Multi-Factor Equation

A possible solution is to model each asset with a multifactor equation:

$$r_i = \mu_i + \beta_{i1} F_1 + ... + \beta_{ik} F_k + \epsilon_i, \tag{4}$$

where

- F_j is the deviation of the random factor j from its mean
- $cov(F_j, F_l) = 0$ for all $j \neq l$
- ϵ_i are asset specific risks with zero expectation, uncorrelated and independent of the factors.

The Multi-Factor Equation

Examples of typical factors:

- inflation
- interest rates
- GDP etc.

Example:

Risk-Return Optimization

The Multi-Factor Equation

The portfolio $r_p = \omega' R$ can be written as

$$r_p = \mu_p + \sum_{j=1}^{k} \beta_{pj} F_j + \epsilon_p,$$

where

$$\mu_p = \omega' \mu, \qquad \beta_{pj} = \sum_{i=1}^{n} \omega_i \beta_{ij}, \qquad \epsilon_p = \sum_{i=1}^{n} \omega_i \epsilon_i.$$

It follows the variance of the portfolio is

$$\sigma_p^2 = \sum_{j=1}^{k} \beta_{pj}^2 \sigma_{F_j}^2 + \sum_{i=1}^{n} \omega_i^2 \sigma_{\epsilon_i}^2.$$

The Multi-Factor Equation

Two sorts of risk:

- systematic or market risk (first term)
- unsystematic risk of the portfolio (second term)

If equal weight is given to each asset, $\omega_i = 1/n$, the unsystematic risk is bounded by c/n for some constant c, so this risk can be diversified away as n grows large!!

Risk-Return Optimization

The Multi-Factor Equation

Using the factor model in the minimum variance optimization problem gives:

$$
\begin{aligned}
\min_{\omega} \quad & \sigma_p^2 = \textstyle\sum_{j=1}^{k} \beta_{pj}^2 \sigma_{F_j}^2 + \sum_{i=1}^{n} \omega_i^2 \sigma_{\epsilon_i}^2 \\
\text{s.t.} \quad & \omega' \mu = \mu_0, \\
& \beta_{pj} = \textstyle\sum_{i=1}^{n} \omega_i \beta_{ij} \\
& \textstyle\sum_{i=1}^{n} \omega_i = 1.
\end{aligned}
$$

The factor sensitivities β_{ij}, factor variances, and specific risk variances can be estimated through linear regression in equation (4).

$\Rightarrow$ significant reduction in the number of parameter estimates needed as compared to optimization problem (3).

Risk-Return Optimization

The Multi-Factor Equation

Example:

Risk-Return Optimization

The Multi-Factor Equation

Remarks:

- Both of the above are quadratic optimization problem.
- A linear optimization problem can be achieved when the variance of the portfolio is replaced with its mean-absolute deviation m_p.
- Since R is multivariate normal, the relation holds that $m_p = \sqrt{\frac{2}{\pi}}\sigma_p$, so minimizing the mean-absolute deviation will produce the **same optimal portfolio** as minimizing the variance.

Elliptical Distributions

Examples of Elliptical Distributions:

Risk-Return Optimization

Elliptical Distributions

The class of elliptical distributions offers special properties in portfolio theory that are useful in minimizing VaR or CVaR:

- For any elliptically distributed random vector R with finite variance for all univariate marginals, variance is equivalent to any positive homogeneous risk measure ρ.
- If $r_p = \omega' R$ and $\tilde{r}_p = \tilde{\omega}' R$ are two linear portfolios with corresponding variances σ_p^2 and $\tilde{\sigma}_p^2$:

$$\rho\left(r_p - \mathrm{E}(r_p)\right) \leq \rho\left(\tilde{r}_p - \mathrm{E}(\tilde{r}_p)\right) \quad \Longleftrightarrow \quad \sigma_p^2 \leq \tilde{\sigma}_p^2.$$

Risk-Return Optimization

Elliptical Distributions

In addition if ρ is translation invariant, the solution to the following risk-return optimization problems coincide:

$$\begin{aligned} \min_{\omega} \quad & \sigma_p^2 \\ \text{s.t.} \quad & r_p = \omega' R, \\ & \mathrm{E}(r_p) = \mu_0, \\ & \textstyle\sum_{i=1}^n \omega_i = 1, \end{aligned} \qquad\qquad \begin{aligned} \min_{\omega} \quad & \rho(r_p) \\ \text{s.t.} \quad & r_p = \omega' R, \\ & \mathrm{E}(r_p) = \mu_0, \\ & \textstyle\sum_{i=1}^n \omega_i = 1, \end{aligned}$$

where μ_0 is the desired return.

Risk-Return Optimization

Elliptical Distributions

Under the assumption assumption of elliptical distributions, the same optimal portfolios will be obtained by:

- minimization of VaR
- minimization of CVaR
- minimization of the variance

But: due to the heavy-tailed nature of financial returns, elliptical distributions may not be an adequate assumption!! See the next section for further details.

4) Optimization Problems, Stochastic Programing and Scenario Analysis

3.1) Single- and Multistage Optimization Problems

3.2) Stochastic Programming

3.3) Scenario Generation

Single- and Multi-Stage Optimization

Single-Stage Optimization

Assume we want to optimize the portfolio according to thr CVaR criterium.

Define

$$X = \left\{ \omega \in \mathbb{R}^n \,\middle|\, \sum_{j=1}^{n} \omega_j = 1, \omega_j \geq 0, j = 1, ..., n \right\}, \tag{5}$$

where $x \in X$ represents the portfolio weights in n assets. The random return on these assets at the end of a time period is represented by $R = (r_1, ..., r_n)'$, and the negative return of the portfolio is given by

$$L(x) = -x'R.$$

Single-Stage Optimization

If the mean of R is given by the vector μ, the risk-return problem is:

$$\min_{x \in X} \mathrm{CVaR}_\beta(x) \quad \text{s.t.} \quad x'\mu \geq \mu_0,$$

where μ_0 is the required portfolio return, and by varying μ_0, the efficient frontier is obtained.

Single- and Multi-Stage Optimization

Single-Stage Optimization

If the uncertainty in the return is given through the set of scenarios $\{R^1, ..., R^S\}$ where each $R^s \in \mathbb{R}^n$ occurs with probability p^s, the problem can be rewritten as:

$$\begin{array}{ll} \min & \zeta + \frac{1}{1-\beta} \sum_{s=1}^{S} p^s \left[-x' R^s - \zeta\right]^+ \\ \text{s.t.} & x'\mu \geq \mu_0, \\ & x \in X, \zeta \in \mathbb{R}, \end{array}$$

Single- and Multi-Stage Optimization

Single-Stage Optimization

By introducing auxiliary variables y^s, $s = 1, ..., S$, a linear program results:

$$\begin{array}{lll} \min & \zeta + \frac{1}{1-\beta} \sum_{s=1}^{S} p^s y^s & \\ \text{s.t.} & x'\mu \geq \mu_0, & \\ & x'R^s + \zeta + y^s \geq 0, & s = 1, ..., S, \\ & y^s \geq 0, & s = 1, ..., S, \\ & x \in X, \zeta \in \mathbb{R}. & \end{array}$$

$\Rightarrow$ This program is used to compare hedging strategies for e.g. international asset allocation.

Multi-Stage Optimization

Extending the single period risk-return problem to a multi-period setting is difficult and some modifications are necessary:

- In a multi-period setting, one usually deals with a **wealth process instead of returns** so that problems will be convex and sometimes linear.
- The general form of a stochastic program with recourse allows **any** portfolio allocation to be made in each stage.
- Typically a function of the wealth process, not the return process, is optimized over the quantities of assets held, not the portfolio weights.
- Instead of risk-return analysis, one can perform risk-reward analysis where the risk, for instance, is a function of the wealth process and the reward is the expected terminal wealth.

Single- and Multi-Stage Optimization

Multi-Stage Optimization

For instance, a multi-period extension of mean-variance analysis could be (see Maranas et al):

$$\max \quad \lambda \mathrm{E}(w_T) - (1-\lambda) var(w_T).$$

Here, w_T is the terminal wealth, and the max is taken over all fixed-mixed decision rules.

Multi-Stage Optimization

Remarks:

- In a so-called fixed-mixed rule, the portfolio is reallocated in each time period to keep a certain percentage of wealth in each asset.
- As λ is varied between zero and one, a type of efficient frontier is obtained.
- While the number of decision variables are greatly reduced, the problem becomes non-convex, and a global search algorithm is necessary.

CVaR and Multi-Stage Optimization

The coherence of a risk measure in a multi-period setting is also defined in terms of a wealth process $w = (w_1, ..., w_T)$ where w_1 is a known deterministic wealth:

Also a weighted average of CVaR over the time horizon is coherent!

If $\text{CVaR}_\beta(-w_t)$ is the CVaR associated with the negative wealth $-w_t$, then a coherent risk measure is given by

$$\rho(w) = \rho(w_1, ..., w_T) = \sum_{t=2}^{T} \mu_t \text{CVaR}_\beta(-w_t), \tag{6}$$

where the weights are nonnegative and sum to one.

CVaR and Multi-Stage Optimization

In this multi-stage setting coherence means that ρ is:

1. convex: $\rho(\lambda w + (1-\lambda)\tilde{w}) \leq \lambda\rho(w) + (1-\lambda)\rho(\tilde{w}), \quad \forall\lambda \in [0,1]$,
2. positive homogeneous: $\rho(\lambda w) = \lambda\rho(w), \quad \forall\lambda \geq 0$,
3. translation invariant: $\rho(w_1 + c, ..., w_T + c) = \rho(w) - c, \quad \forall c \in \mathbb{R}$, and
4. monotonous: if $w_t \leq \tilde{w}_t$ a.s. for $t = 1, ..., T$, then $\rho(w) \geq \rho(\tilde{w})$.

Introduction

Stochastic programming offers a framework that can incorporate many of the characteristics of an ALM problem.

Organisation of the Section:

- General Setup
- Scenario Generation
- The T-Stage ALM Problem
- Application

General Recourse Problem

A 2-Stage Recourse Problem

In a 2-stage recourse problem, a recourse decision is made after a realization of uncertainty.

Consider the following asset allocation problem:

- *First Stage Decision:* the initial portfolio allocation
- *Uncertainty:* the asset returns
- *Recourse Decision:* the portfolio adjustments.

$\longrightarrow$ This 2-stage recourse problem finds the optimal initial and rebalanced allocations for the given distribution of future stock movements.

General Recourse Problem

A 2-Stage Recourse Problem

- The first stage has a vector of initial decisions $x_1 \in \mathbb{R}^{n_1}$ made at $t = 1$ when there is a known distribution of future uncertainty.
- The second stage decisions $x_2 \in \mathbb{R}^{n_2}$ adapt at $t = 2$ after the first stage uncertainty ξ_1 is realized.
- The second stage decisions usually also consider the distribution of future uncertainty ξ_2 realized after $t = 2$.

General Recourse Problem

Mathematical Description

This setup is described mathematically by first considering how the optimal recourse decision is determined. We define:

- x_1 the stage decision vector
- ξ_1 and a given realization of the first stage uncertainty
- $q_2(x_1, x_2, \xi_1)$ is a cost of decision x_2 for the given realization of the first stage uncertainty ξ_1 and the given first stage decision x_1,
- $Q_2(x_1, x_2, \xi_1, \xi_2)$ is the cost of decision x_2 for given realizations of uncertainties ξ_1 and ξ_2 and the given first stage decision x_1,
- $B_2(\xi_1)$ is the *technology matrix* that converts a first stage decision into resources in the second stage, and
- $A_2(\xi_1)$ is the *recourse matrix*.

General Recourse Problem

Mathematical Description

Then the best recourse decision is found through the following second stage problem:

$$
\begin{aligned}
\min_{x_2} \quad & q_2(x_1, x_2, \xi_1) + \mathrm{E}_{\xi_2}\left(Q_2(x_1, x_2, \xi_1, \xi_2) \middle| \, \xi_1 \right) \\
\text{s.t.} \quad & B_2(\xi_1)x_1 + A_2(\xi_1)x_2 = b_2(\xi_1), \\
& l_2(\xi_1) \leq x_2 \leq u_2(\xi_1)
\end{aligned} \tag{7}
$$

General Recourse Problem

Remarks:

- It is possible to remove the cost function Q_2 by including the second term of the objective in the cost function q_2.
- The problem is said to have *fixed recourse* when A_2 is independent of ξ_1.
- The subscripts indicate at which t a value is known except in the case of ξ_t
- For instance, the realizations of B_2, A_2, and b_2 are all known at $t = 2$, which is the beginning of the second stage, but ξ_2 is not realized until after $t = 2$.

General Recourse Problem

The full 2-stage recourse problem

The second stage problem as incorporated as follows:
With the optimal value of the second stage problem (7) denoted by $Q_1(x_1, \xi_1)$, the 2-stage problem minimizes:

- the sum of a first stage cost $q_1(x_1)$ and
- the expected value of the second stage cost $\mathrm{E}Q_1(x_1, \xi_1)$:

$$\begin{aligned} \min_{x_1} \quad & q_1(x_1) + \mathrm{E}Q_1(x_1, \xi_1) \\ \text{s.t.} \quad & A_1 x_1 = b_1, \\ & l_1 \le x_1 \le u_1. \end{aligned} \tag{8}$$

General Recourse Problem

2-Stage and Multistage Problems

- only one recourse decision can be made, instead of a sequence of decisions over the time horizon
- a multistage recourse program can provide a more realistic model, but it is more complex and can often be very difficult to solve numerically
- as in the 2-stage problem, the initial vector of decisions x_1 is made before the first realization of uncertainty ξ_1, and a second stage decision x_2 is then made based on x_1 and ξ_1
- in the T-stage problem, this process continues for the uncertainties ξ_t, $t = 1, ..., T - 1$, and the decisions vectors x_t, $t = 1, ..., T$
- there is usually one additional realization of uncertainty ξ_T following the final decision x_T

General Recourse Problem

The Multistage Problem

The T-stage recourse program can be defined recursively as an extension of the 2-stage program:

- $\xi^t = \{\xi_j, j = 1, ..., t\}$ denotes the uncertainty up to and including stage t, for $t = 1, ..., T$, where each ξ_j is the uncertainty realized in stage j.
- $x^t = \{x_j, j = 1, ..., t\}$ denotes the decisions up to and including stage t, where each x_j is the decision made for stage j.

Then the first stage problem is essentially the same as problem (8):

$$\begin{aligned} \min_{x_1} \quad & q_1(x_1) + \mathrm{E}_{\xi_1} Q_1(x^1, \xi^1) \\ \text{s.t.} \quad & A_1 x_1 = b_1, \\ & l_1 \leq x_1 \leq u_1, \end{aligned} \tag{9}$$

General Recourse Problem

The Multistage Problem

Q_t, for $t = 1, ..., T-1$ is given by the minimization problems:

$$\begin{aligned} Q_t(x^t, \xi^t) = \quad \min_{x_{t+1}} \quad & q_{t+1}(x^{t+1}, \xi^t) + \mathrm{E}_{\xi_{t+1}} \left(Q_{t+1}(x^{t+1}, \xi^{t+1}) \middle| \, \xi^t \right) \\ \text{s.t.} \quad & B_{t+1}(\xi^t) x_t + A_{t+1}(\xi^t) x_{t+1} = b_{t+1}(\xi^t), \\ & l_{t+1}(\xi^t) \le x_{t+1} \le u_{t+1}(\xi^t), \end{aligned} \tag{10}$$

and $Q_T(x^T, \xi^T)$ is a known function, not the solution to another minimization problem.

General Recourse Problem

The Multistage Problem

- it is possible to set $Q_T = 0$ by including the second term of the objective in q_T.
- the above problem (9-10) is a form of the multistage recourse problem that is relevant to the ALM problem that will be presented soon
- alternative forms, such as that found in [**?**], allow the first constraint of (10) to depend on all decisions up to time t:

$$\sum_{\tau=1}^{t} B_{t+1,\tau}(\xi^t)x_\tau + A_{t+1}(\xi^t)x_{t+1} = b_{t+1}(\xi^t), \tag{11}$$

Scenario Generation Techniques

Introduction

To numerically solve the recourse problem (9-10), often so-called **scenario generation techniques** are used. The distribution of $(\xi_1, ..., \xi_T)$ is approximated by a set of scenarios:

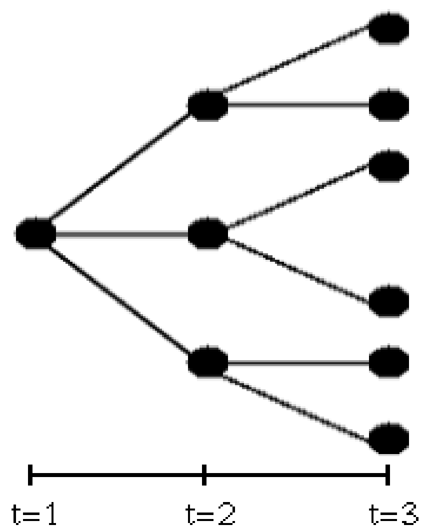

Figure 1: An exemplary Scenario Tree.

Scenario Generation Techniques

Scenario Trees

- a first stage optimal allocation is found in the node at $t = 1$
- optimal recourse allocations are found in every node at $t = 2$
- in a 2-stage problem, there is no additional allocation decision made at the nodes at $t = 3$
- the tree shown in the figure is called *balanced* because each node at $t = 2$ is connected to two nodes at $t = 3$.

Scenario Generation Techniques

Scenario Trees

- the nodes of the scenario tree are numbered starting with the value of one at $t = 1$
- let I_t be the number of nodes up to and including those at t
- sets of indices $\mathcal{I}_t = \{I_{t-1} + 1, ..., I_t\}$, are defined for $t = 2, ..., T + 1$, with $I_1 = 1$
- a scenario s, which is a path through the scenario tree, is then represented by the set of indices $(i_2, ..., i_{T+1})$ where $i_t \in \mathcal{I}_t$

Scenario Trees

Predecessor and Descendant:

Two useful functions defined on the node indices are the predecessor, $pred(\cdot)$, and the descendant, $dec(\cdot)$:

- $pred(i_t)$ returns the node in $\mathcal{I}_{t-1}$ connected to i_t
- $dec(i_t)$ returns a subset of nodes in $\mathcal{I}_{t+1}$ connected to i_t
- at t, the probability of being at node $i_t \in \mathcal{I}_t$ is denoted by $p(i_t)$ so that $\sum_{i_t \in \mathcal{I}_t} p(i_t) = 1$
- sometimes it is more useful to use the transition probabilities $p(i_t, i_{t+1})$, for $i_{t+1} \in dec(i_t)$ where $\sum_{i_{t+1} \in dec(i_t)} p(i_t, i_{t+1}) = 1$.

Scenario Generation Techniques

Scenario Trees

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Scenario Generation Techniques

Importance Sampling

One technique in scenario selection is sequential importance sampling:

- general idea: obtain scenarios that are important (in some sense) in the stochastic program
- sequential importance sampling obtains these scenarios in an iterative fashion (see e.g. Dupacova, 2000):
  1. generate scenarios for a for a given tree structure
  2. solve the stochastic program is to obtain values for the importance sampling criterion at each node.
  3. use nodal values to determine where the structure of the scenario tree should be changed and/or where to resample a subtree

Scenario Generation Techniques

Importance Sampling - Discretization

Discretization is an alternative to sampling from a distribution:

- a relatively simple technique for discretization is **moment matching** (see e.g. Dupacova, 2000)
- to discretize the normal distribution it is possible to match the first two moments with three symmetric points

Example:

Scenario Generation Techniques

Importance Sampling - Discretization

Alternative discretization techniques rely on the minimization of transportation metrics to approximate a continuous distribution with a discrete distribution:

- assume that a desired scenario tree structure has already been determined
- the goal is to minimize the difference between the optimal value of the stochastic program with the true distribution and the optimal value of the stochastic program with the approximate distribution

Example:

Scenario Generation Techniques

Generating Sample Paths of Uncertain Data

There are many different methods to generate sample paths of the uncertain data. Sample paths may come from

- expert's expectation
- historical observations
- time-series model

The remaining problem is then to convert a set of sample paths into a scenario tree.

Generating Sample Paths of Uncertain Data

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The remaining problem is then to convert a set of sample paths into a scenario tree. This can be done e.g. by clustering similar first stage values of the same paths.

Scenario Generation Techniques

Deterministic Equivalent Forms

The discrete and finite distribution of a scenario tree allows the stochastic recourse problem to be written as a deterministic program.

Once a scenario tree is constructed, each node i_t of the scenario tree determines values for $A_t(\xi^{t-1})$, $B_t(\xi^{t-1})$, $b_t(\xi^{t-1})$, $l_t(\xi^{t-1})$, $u_t(\xi^{t-1})$, and $q_t(\cdot, \xi^{t-1})$ which are denoted by A_{i_t}, B_{i_t}, b_{i_t}, l_{i_t}, u_{i_t}, and $q_{i_t}(\cdot)$.

Deterministic Equivalent Forms

The recourse problem (9-10) can then be written as

$$\begin{aligned} \min_{x_1} \quad & q_1(x_1) + \sum_{i_2 \in \mathcal{I}_2} p(i_2) Q_{i_2}(x^1) \\ \text{s.t.} \quad & A_1 x_1 = b_1, \\ & l_1 \le x_1 \le u_1, \end{aligned} \tag{12}$$

with Q_{i_t}, for $i_t \in \mathcal{I}_t$, $t = 2, ..., T$, given by the minimization problems

$$\begin{aligned} Q_{i_t}(x^{t-1}) = \quad \min_{x_t} \quad & q_{i_t}(x^t) + \sum_{i_{t+1} \in dec(i_t)} p(i_t, i_{t+1}) Q_{i_{t+1}}(x^t) \\ \text{s.t.} \quad & B_{i_t} x_{t-1} + A_{i_t} x_t = b_{i_t}, \\ & l_{i_t} \le x_t \le u_{i_t}, \end{aligned} \tag{13}$$

and $Q_{i_{T+1}}$ can be taken to be equal to zero.

Conversion into a fully linear problem

Alternatively, by introducing auxiliary variables, the piecewise linear problem can be converted into a fully linear problem (with potentially a huge number of decision variables).

In this case, the $q_{i_t}(\cdot)$ will take the linear form:

$$q_{i_t}(\cdot) = \langle q_{i_t}, \cdot \rangle, \tag{14}$$

where q_{i_t} is now a vector of appropriate dimension.

Conversion into a fully linear problem

The deterministic equivalent for the linear program in arborescent form carefully considers the structure of the scenario tree:

$$
\begin{aligned}
&\min \quad \langle q_1, x_1\rangle + \textstyle\sum_{i_2\in\mathcal{I}_2} p(i_2)\langle q_{i_2}, x_{i_2}\rangle + \cdots + \textstyle\sum_{i_T\in\mathcal{I}_T} p(i_T)\langle q_{i_T}, x_{i_T}\rangle \\
&\text{subject to} \\
&\begin{array}{lclll}
A_1 x_1 & & = & b_1, & \\
B_{i_2} x_1 + \quad A_{i_2} x_{i_2} & & = & b_{i_2}, & \forall i_2 \in \mathcal{I}_2, \\
 & & \vdots & & \\
 & B_{i_T} x_{pred(i_T)} + A_{i_T} x_{i_T} & = & b_{i_T}, & \forall i_T \in \mathcal{I}_T,
\end{array} \\
&l_{i_t} \le x_{i_t} \le u_{i_t}, \qquad \forall i_t \in \mathcal{I}_t, \qquad t = 1, ..., T.
\end{aligned}
\tag{15}
$$

Conversion into a fully linear problem

- this arborescent form implicity includes non-anticipatory constraints that the decision taken at t does not depend on the uncertainty that is realized in the future.
- the decision vectors are x_{i_t}, $i_t \in \mathcal{I}_t$, $t = 1, ..., T$, so there is one decision for each node of the scenario tree except for those at $T + 1$.

Scenario Generation Techniques

The split-variable formulation

Assume there are S sample paths in the scenario tree such that S independent subproblems are created by allowing all decisions to be scenario dependent:

Then the individual subproblem for scenario s with nodes $(i_2, ..., i_{T+1})$ is

$$\begin{array}{llcl} \min & \langle q_1, x_1^s \rangle + \langle q_{i_2}, x_2^s \rangle + ... + \langle q_{i_T}, x_T^s \rangle & & \\ & & & \\ \text{s.t.} & A_1 x_1^s & = & b_1, \\ & B_{i_2} x_1^s + \quad A_{i_2} x_2^s & = & b_{i_2}, \\ & & \vdots & \\ & \qquad\qquad B_{i_T} x_{T-1}^s + A_{i_T} x_T^s & = & b_{i_T}, \end{array} \tag{16}$$

plus any upper and lower bounds on x_t^s.

Scenario Generation Techniques

The split-variable formulation

When combining all subproblems into one problem, non-anticipatory constraints must be explicitly considered in this formulation

For any two scenarios s and s' with a common path up to and including t, $x_j^s = x_j^{s'}$, for $j = 1, ..., t$, must be enforced.

If p^s is the probability of scenario s, the overall split-variable representation for the multistage program is

$$\min \qquad \sum_{s=1}^{S} p^s \left(\langle q_1, x_1^s \rangle + \langle q_{i_2}, x_2^s \rangle + ... + \langle q_{i_T}, x_T^s \rangle \right),$$

subject to a set of constraints (16) for each s, the non-anticipatory constraints, and any upper and lower bound constraints on x_t^s.