The Modified Tempered Stable Distribution, GARCH Models and Option Pricing

Young Shin Kim, Svetlozar T. Rachev *,
Dong Myung Chung and Michele Leonardo Bianchi †

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Young Shin Kim
Department of Statistics, Econometrics and Mathematical Finance, School of Economics and Business Engineering, University of Karlsruhe and KIT
Kollegium am Schloss, Bau II, 20.12, R210, Postfach 6980, D-76128, Karlsruhe, Germany
E-mail: aaron.kim@statistik.uni-karlsruhe.de

Svetlozar T. Rachev
Chair-Professor, Chair of Statistics, Econometrics and Mathematical Finance, School of Economics and Business Engineering, University of Karlsruhe and KIT, and Department of Statistics and Applied Probability, University of California, Santa Barbara, and Chief-Scientist, FinAnalytica INC
Kollegium am Schloss, Bau II, 20.12, R210, Postfach 6980, D-76128, Karlsruhe, Germany
E-mail: rachev@statistik.uni-karlsruhe.de

Dong Myung Chung
Department of Mathematics,The Catholic University of Korea
Yeokgok 2-dong 43-1, Wonmi-gu, Bucheon-si, Gyeonggi-do 420-743, Korea
E-mail: dmchung@catholic.ac.kr

Michele Leonardo Bianchi
Department of Mathematics, Statistics, Computer Science and Applications, University of Bergamo
Via dei Caniana 2, I-24127, Bergamo, Italy
E-mail:michele-leonardo.bianchi@unibg.it

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Abstract

We introduce a new variant of the tempered stable distribution, named the modified tempered stable (MTS) distribution and we develop a GARCH option pricing model with MTS innovations. This model allows the description of some stylized empirical facts observed in financial markets, such as volatility clustering, skewness, and heavy tails of stock returns. To demonstrate the advantages of the MTS-GARCH model, we present the results of the parameter estimation.
Key words: option pricing, GARCH process, tempered stable distribution, volatility clustering

1 Introduction

Since Black and Scholes [2] introduced the pricing and hedging theory for the option market, their model has been the most popular model for option pricing. However, the model which assumes homoskedasticity and lognormality, cannot explain stylized empirical facts such as skewness, heavy tails, and volatility clustering of stock returns.

To explain these empirical facts, Mandelbrot [12, 13] was the first to use a non-normal Lévy process as an asset price process. Hurst, Platen and Rachev [9] used a model based on stable processes to price options. However, stable distributions have infinite moments of the second or higher order because of the heavy distributional tails. To have more adaptability, the class of tempered stable processes has been introduced under different names including: "truncated Lévy flight" (Koponen [11]), "KoBoL" process (Boyarchenko and Levendorskiǐ [3]), and "CGMY" process (Carr et al. [4]). Rosiński [17] generalized the notion of tempered stable processes. In his extension, tempered stable processes are characterized by the spectral (Rosiński) measure. Moreover, several concrete subclasses of the generalized tempered stable distributions and related Ornstein-Uhlenbeck processes have been presented in [20]. By assuming a Markovian stock return process and by considering the generalized Fourier transform, Carr et al. [4] obtained a close form solution to price European options. However, the Markov property is often rejected by the empirical evidence as in the case in which stock returns exhibit volatility clustering.

The GARCH option pricing models have been developed to price options under the assumption of volatility clustering. GARCH models of Duan [6], Heston and Nandi [8] are remarkable works on the non-Markovian structure of asset returns even though they did not take into account conditional leptokurtosis and skewness. Duan et al. [7] modified the classical GARCH model by adding jumps to the innovation processes. Furthermore, Menn and Rachev [15, 16] introduced an enhanced GARCH model with innovations which follow the smoothly truncated stable (STS) distribution; it also has a finite variance and at the same time allows for conditional leptokurtosis and skewness.

In this paper, we introduce a variant of the tempered stable distributions, called a modified tempered stable (MTS) distribution, and apply it to the GARCH option pricing model.

The MTS distribution is obtained by taking an α-stable law and multiplying the Lévy measure by a modified Bessel function of the second kind onto each half of the real axis. It is infinitely divisible, has a closed form characteristic function, finite moments of all orders. Its Lévy measure behaves asymptotically like the α-stable distribution near zero and has exponential decay of the tails. We can show that MTS distribution is not included in the class of Rosiński's tempered stable distributions, but has properties similar to the tempered stable distributions.

The GARCH option pricing model presented in this paper follows the method introduced by Menn and Rachev [15, 16]. However, instead of STS innovations, we assume that the innovations of the classical GARCH model follow the MTS distribution with zero mean and unit variance, and we are able to describe both leptokurtosis and skewness. In contrast to the STS distribution, the Laplace transform of a MTS distribution is analytic, therefore it is more tractable. Moreover, it is infinitely divisible and its characteristic function provides a concrete method to find an equivalent martingale measure by applying a general result on density transformations for Levy processes, presented by Sato [19].

The remainder of this paper is organized as follows: Section 2 introduces the MTS distribution. The characteristic function, the cumulant, and asymptotic behavior of the MTS distribution are presented in the first subsection, followed by measure changes of the MTS distributions. The GARCH model with MTS innovations and its empirical investigations are reported in the third section. Section 4 is a summary of our conclusions. Proofs are presented in the Appendix.

2 The Model

2.1 Tempered Stable Distributions

Before introducing the MTS distribution and the MTS-GARCH model, let us review the tempered stable distribution. It is well known that α-stable distributions have infinite p-th moments for all $p \geq \alpha$. This is due to the fact that its Lévy density decays polynomially. Tempering of the tails with the exponential rate is one choice to ensure finite moments. The Tempered Stable (TS) distribution is obtained by taking a symmetric α-stable distribution and multiplying the Lévy measure with exponential functions on each half of the real axis. Indeed, it is defined in the following:

Definition 2.1. *An infinitely divisible distribution is called a* tempered stable (TS) distribution *with parameter* $(C_1, C_2, \lambda_+, \lambda_-, \alpha)$*, if its Lévy triplet* (σ^2, ν, γ) *is given by* $\sigma = 0$, $\gamma \in \mathbb{R}$ *and*

$$\nu(dx) = \left(\frac{C_1 e^{-\lambda_+ x}}{x^{1+\alpha}} 1_{x>0} + \frac{C_2 e^{-\lambda_- |x|}}{|x|^{1+\alpha}} 1_{x<0} \right) dx, \tag{2.1}$$

where $C_1, C_2, \lambda_+, \lambda_- > 0$ *and* $\alpha < 2$.

This process was first introduced by Koponen [11] under the name of Truncated Lévy Flights. In particular, if $C_1 = C_2 = C > 0$, then this distribution is called the CGMY distribution which has been used in Carr et al. [4] for financial modeling. In the above definition, λ_+ and λ_- give the tail decay rates,

α describes the jumps near zero, and C_1 and C_2 determine the arrival rate of jumps for a given size.

The characteristic function ϕ_{TS} for a tempered stable distribution is given by

$$\begin{aligned}\phi_{TS}(u) = \exp(iu\mu + C_1\Gamma(-\alpha)((\lambda_+ - iu)^\alpha - \lambda_+^\alpha) \\ + C_2\Gamma(-\alpha)((\lambda_- + iu)^\alpha - \lambda_-^\alpha)),\end{aligned} \tag{2.2}$$

for some $\mu \in \mathbb{R}$. Moreover, ϕ_{TS} can be extended to the region $\{z \in \mathbb{C} : \mathrm{Im}(z) \in (-\lambda_-, \lambda_+)\}$. The proof can be found in [4, 5, 10]. Using the characteristic function, we obtain cumulants

$$c_m(X) = \frac{d^m}{du^m} \log \phi_{TS}(u)\Big|_{u=0}$$

of all orders.

Proposition 2.2. *Let X be a tempered stable distributed random variable whose characteristic function is given by (2.2). The cumulant $c_n(X)$ of X is given by*

$$c_n(X) = \Gamma(n-\alpha)C_1\lambda_+^{\alpha-n} + (-1)^n\Gamma(n-\alpha)C_2\lambda_-^{\alpha-n}, \quad \textit{for} \quad n \in \mathbb{N}, \quad n \geq 2,$$

and $c_1(X) = \mu + \Gamma(1-\alpha)C_1\lambda_+^{\alpha-1} - \Gamma(1-\alpha)C_2\lambda_-^{\alpha-1}$.

2.2 Rosiński's Generalization of Tempered Stable Distributions

In this section we will review the definition of the generalized tempered stable distributions introduced by Rosiński [17]. Let the Lévy measure M_0 of an α-stable distribution on $\mathbb{R}^d$ in polar coordinates be of the form

$$M_0(dr, du) = r^{-\alpha-1} dr\, \sigma(du) \tag{2.3}$$

where $\alpha \in (0, 2)$ and σ is a finite measure on S^{d-1}. A (generalized) tempered α-stable distribution is defined by tempering the radial term of M_0 as follows:

Definition 2.3 (Definition 2.1. in [17])**.** *Let $\alpha \in (0,2)$ and σ be a finite measure on S^{d-1}. A probability measure on $\mathbb{R}^d$ is called* tempered α-stable (denoted as TαS) *if is infinitely divisible without Gaussian part and whose Lévy measure M can be written in polar coordinates as*

$$M(dr, du) = r^{-\alpha-1} q(r,u) dr\, \sigma(du). \tag{2.4}$$

where $q : (0,\infty) \times S^{d-1} \mapsto (0,\infty)$ is a Borel function such that $q(\cdot, u)$ is completely monotone with $q(\infty, u) = 0$ for each $u \in S^{d-1}$. A TαS distribution is called a proper TαS *distribution if* $\lim_{r\to 0^+} q(r,u) = 1$ *for each $u \in S^{d-1}$.*

The completely monotonicity of $q(\cdot, u)$ means that $(-1)^n \frac{d}{dr} q(r,u) > 0$ for all $r > 0$, $u \in S^{d-1}$, and $n = 0, 1, 2, \cdots$.

TαS distributions are characterized by the *spectral measure* or *Rosiński measure* defined in Definition 2.4 in [17]. Moreover, [17] presents the characteristic function, short and long time behavior, absolute continuity, and shot-noise-type series representation for TαS distributions and Lévy processes induced by the TαS distributions.

2.3 The Modified Tempered Stable Distributions

In this section, we introduce a variant of the tempered stable distribution named Modified Tempered Stable (MTS) distribution. The MTS distribution is defined as follows:

Definition 2.4. *An infinitely divisible distribution is said to be a* modified tempered stable (MTS) *distribution if its Lévy triplet is given by*

$$\begin{aligned}
&\sigma^2 = 0 \\
&\nu(dx) = C\left(\frac{\lambda_+^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_+ x)}{x^{\frac{\alpha+1}{2}}} 1_{x>0} + \frac{\lambda_-^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_- |x|)}{|x|^{\frac{\alpha+1}{2}}} 1_{x<0} \right) dx \\
&\quad \gamma = \mu + C\left(\frac{\Gamma\left(\frac{1-\alpha}{2}\right)}{2^{\frac{\alpha+1}{2}}} \left(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}\right) - \lambda_+^{\frac{\alpha-1}{2}} K_{\frac{\alpha-1}{2}}(\lambda_+) + \lambda_-^{\frac{\alpha-1}{2}} K_{\frac{\alpha-1}{2}}(\lambda_-) \right),
\end{aligned}$$

where $C > 0$, $\lambda_+, \lambda_- > 0$, $\mu \in \mathbb{R}$, $\alpha \in (-\infty, 2) \setminus \{1\}$ and $K_p(x)$ is the modified Bessel function of the second kind (See [1, p.290]). We denote an MTS distributed random variable X by $X \sim MTS(\alpha, C, \lambda_+, \lambda_-, \mu)$. The Lévy measure $\nu(dx)$ is called the MTS Lévy measure with parameter $(\alpha, C, \lambda_+, \lambda_-)$.

The MTS distribution is obtained by taking a symmetric α-stable distribution with $\alpha \in (0,2)$ and multiplying the Lévy measure with $(\lambda|x|)^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda|x|)$ on each half of the real axis. The measure can be extended to the case of $\alpha \leq 0$. If $\alpha = 1$, then γ may not be defined. Hence, we remove it. The following result shows that $\nu(dx)$ is a Lévy measure.

Proposition 2.5. *Let ν be a Borel measure on $\mathbb{R}$ such that $\nu(0) = 0$ and*

$$\nu(dx) = C\left(\frac{\lambda_+^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_+ x)}{x^{\frac{\alpha+1}{2}}} 1_{x>0} + \frac{\lambda_-^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_-|x|)}{|x|^{\frac{\alpha+1}{2}}} 1_{x<0}\right) dx, \tag{2.5}$$

where $C > 0$, $\lambda_+, \lambda_- > 0$, and $\alpha < 2$. Then the measure ν is a Lévy measure on $\mathbb{R}$.

The next result follows from (A.2) and (A.3) in the Appendix.

Proposition 2.6. *Let*

$$f(x) = C\left(\frac{\lambda_+^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_+ x)}{x^{\frac{\alpha+1}{2}}} 1_{x>0} + \frac{\lambda_-^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_-|x|)}{|x|^{\frac{\alpha+1}{2}}} 1_{x<0}\right),$$

where $C > 0$, $\lambda_+, \lambda_- > 0$ and $\alpha \in (0,2) \setminus \{1\}$. Then

$$f(x) \sim 2^{\frac{\alpha-1}{2}} C\Gamma\left(\frac{\alpha+1}{2}\right) \frac{1}{x^{\alpha+1}}, \qquad \text{as } x \to 0, \tag{2.6}$$

$$f(x) \sim \sqrt{\frac{\pi}{2}} C \lambda_+^{\frac{\alpha}{2}} \frac{e^{-\lambda_+ x}}{x^{\frac{\alpha}{2}+1}}, \qquad \text{as } x \to \infty, \tag{2.7}$$

$$f(x) \sim \sqrt{\frac{\pi}{2}} C \lambda_-^{\frac{\alpha}{2}} \frac{e^{-\lambda_- |x|}}{|x|^{\frac{\alpha}{2}+1}}, \qquad \text{as } x \to -\infty. \tag{2.8}$$

Remark 2.7. *The Lévy measures of MTS distribution behaves like α-stable distribution near zero and decreases exponentially with rates λ_+ and λ_- at the tails.*

The Lévy measure ν of the MTS distribution can be reformed in polar coordinates as

$$\nu(dx) = M_{\text{MTS}}(dr, du) = r^{-\alpha-1} q_{\text{MTS}}(r,u) dr\, \sigma(du),$$

where σ is a finite measure on $S^0 = \{-1, 1\}$ such that

$$\sigma(\{1\}) = \sigma(\{-1\}) = 2^{\frac{\alpha-1}{2}} C\Gamma\left(\frac{\alpha+1}{2}\right),$$

and the polar coordinate function $q_{\text{MTS}} : (0,\infty) \times S^0 \mapsto (0,\infty)$ is given by

$$q_{\text{MTS}}(r,u) = \begin{cases} 2^{\frac{1-\alpha}{2}} \left(\Gamma\left(\frac{\alpha+1}{2}\right)\right)^{-1} (\lambda_+ r)^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_+ r), & u = 1 \\ 2^{\frac{1-\alpha}{2}} \left(\Gamma\left(\frac{\alpha+1}{2}\right)\right)^{-1} (\lambda_- r)^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_- r), & u = -1 \end{cases}. \tag{2.9}$$

The MTS distribution is not in the class of the tempered α stable distribution generalized by Rosiński, while $M_{\text{MTS}}(dr, du)$ looks like equation (2.4). Indeed, $q_{\text{MTS}}(\infty, u) = 0$ and $\lim_{r\to 0^+} q_{\text{MTS}}(r,u) = 1$, but $\frac{\partial^2}{\partial r^2} q_{\text{MTS}}(r,u)$ is not always positive. Figure 1 shows the graph of $y = \frac{\partial^2}{\partial r^2} q_{\text{MTS}}(r,1)$ provided that $\lambda_+ = 1$ and $\alpha = 1.5$. We can show that $\frac{\partial^2}{\partial r^2} q_{\text{MTS}}(r,1) < 0$ if $0 < r < 1$. It means, $q_{\text{MTS}}(\cdot, u)$ is not completely monotone, and hence the MTS distribution does not satisfy the condition of complete monotonicity in Definition 2.3.

The characteristic function of the MTS distribution is given in the following result.

Theorem 2.8. *Let $X \sim MTS(\alpha, C, \lambda_+, \lambda_-, \mu)$. Then the characteristic function of X is given by*

$$\phi_X(u; \alpha, C, \lambda_+, \lambda_-, \mu) = \exp(iu\mu + G_R(u; \alpha, C, \lambda_+, \lambda_-) + G_I(u; \alpha, C, \lambda_+, \lambda_-)),$$

where for $u \in \mathbb{R}$,

$$\begin{aligned} &G_R(u; \alpha, C, \lambda_+, \lambda_-) \\ &= \begin{cases} \sqrt{\pi} 2^{-\frac{\alpha}{2}-\frac{3}{2}} C\Gamma(-\frac{\alpha}{2}) \left((\lambda_+^2 + u^2)^{\frac{\alpha}{2}} - \lambda_+^\alpha + (\lambda_-^2 + u^2)^{\frac{\alpha}{2}} - \lambda_-^\alpha\right) & \text{if } \alpha \neq 0 \\ \sqrt{\pi} 2^{-\frac{3}{2}} C \left(\log\left(\frac{\lambda_+^2}{\lambda_+^2 + u^2}\right) + \log\left(\frac{\lambda_-^2}{\lambda_-^2 + u^2}\right)\right) & \text{if } \alpha = 0 \end{cases} \end{aligned}$$

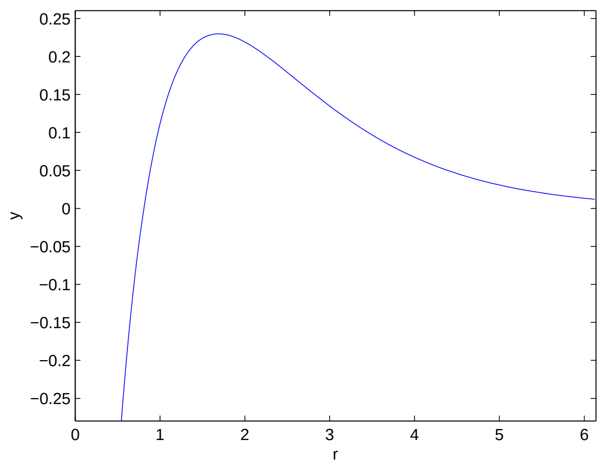

Figure 1: Graph of $y = \frac{\partial^2}{\partial r^2} q_{\mathrm{MTS}}(r,1)$ where $\lambda_+ = 1$ and $\alpha = 1.5$.

and

$$
\begin{aligned}
&G_I(u;\alpha,C,\lambda_+,\lambda_-)\\
&= \frac{iuC\Gamma\left(\frac{1-\alpha}{2}\right)}{2^{\frac{\alpha+1}{2}}}\left(\lambda_+^{\alpha-1}F\left(1,\frac{1-\alpha}{2};\frac{3}{2};-\frac{u^2}{\lambda_+^2}\right)-\lambda_-^{\alpha-1}F\left(1,\frac{1-\alpha}{2};\frac{3}{2};-\frac{u^2}{\lambda_-^2}\right)\right),
\end{aligned}
$$

where F is the hypergeometric function See [1, p.361]. Moreover, ϕ_X can be extended to the region $\{z \in \mathbb{C} : |\mathrm{Im}(z)| < \lambda_+ \wedge \lambda_-\}$.

Corollary 2.9. *Let $X \sim MTS(\alpha, C, \lambda_+, \lambda_-; \mu)$. Then the Laplace transform of X is given by*

$$
\begin{aligned}
&E[\exp(uX)] \\
&= \exp\left(u\mu + G_R(-iu;\alpha,C,\lambda_+,\lambda_-) + G_I(-iu;\alpha,C,\lambda_+,\lambda_-)\right)
\end{aligned} \tag{2.10}
$$

for $u \in \mathbb{C}$ with $|\mathrm{Re}(u)| < \lambda_+ \wedge \lambda_-$.

Using the characteristic function, we obtain the cumulants of all orders.

Proposition 2.10. *Let $X \sim MTS(\alpha, C, \lambda_+, \lambda_-, \mu)$ with $\alpha \in (-\infty, 1) \setminus \{\frac{1}{2}\}$.*

The cumulants $c_m(X)$ of X are given as follows :

(2.11)

$$c_m(X) = \begin{cases} \mu + 2^{-\frac{\alpha+1}{2}} C\Gamma\left(\frac{1-\alpha}{2}\right)\left(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}\right) & \text{if } m = 1 \\ 2^{m-\frac{\alpha+3}{2}} \left(\dfrac{m-1}{2}\right)! C\Gamma\left(\dfrac{m-\alpha}{2}\right)\left(\lambda_+^{\alpha-m} - \lambda_-^{\alpha-m}\right) & \text{if } m = 3, 5, 7, \cdots \\ 2^{-\frac{\alpha+3}{2}} \sqrt{\pi} \dfrac{m!}{\left(\frac{m}{2}\right)!} C\Gamma\left(\dfrac{m-\alpha}{2}\right)\left(\lambda_+^{\alpha-m} + \lambda_-^{\alpha-m}\right) & \text{if } m = 2, 4, 6 \cdots \end{cases}$$

Remark 2.11. *Let $X \sim MTS(\alpha, C, \lambda_+, \lambda_-, \mu)$.*

1. *By Proposition 2.10, we obtain the mean, variance, skewness and excess kurtosis of X which are given as follows :*

$$E[X] = c_1(X) = \mu + 2^{-\frac{\alpha+1}{2}} C\Gamma\left(\frac{1-\alpha}{2}\right)\left(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}\right)$$

$$\mathrm{Var}(X) = c_2(X) = 2^{-\frac{\alpha+1}{2}} \sqrt{\pi} C\Gamma\left(1 - \frac{\alpha}{2}\right)\left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}\right)$$

$$s(X) = \frac{c_3(X)}{c_2(X)^{\frac{3}{2}}} = \frac{2^{\frac{\alpha+9}{4}} \Gamma\left(\frac{3-\alpha}{2}\right)\left(\lambda_+^{\alpha-3} - \lambda_-^{\alpha-3}\right)}{\pi^{\frac{3}{4}} C^{\frac{1}{2}} \left(\Gamma\left(1-\frac{\alpha}{2}\right)\left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}\right)\right)^{\frac{3}{2}}}$$

$$k(X) = \frac{c_4(X)}{c_2(X)^2} = \frac{3 \cdot 2^{\frac{\alpha+3}{2}} \Gamma\left(2-\frac{\alpha}{2}\right)\left(\lambda_+^{\alpha-4} + \lambda_-^{\alpha-4}\right))}{\sqrt{\pi} C \left(\Gamma\left(1-\frac{\alpha}{2}\right)\left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}\right)\right)^2}.$$

2. *Figure 2 illustrates the dependence of skewness $s(X)$ and excess kurtosis $k(X)$ on λ_+ and λ_- when α and C are fixed.*

3. *λ_+ and λ_- control the rate of decay on the positive and negative part, respectively. If $\lambda_+ > \lambda_-$ ($\lambda_+ < \lambda_-$), then the distribution is skewed to the left (right). Moreover, if $\lambda_+ = \lambda_-$, then it is symmetric. Figure 3 illustrates this fact.*

4. *C controls the kurtosis of the distribution. If C increases, then the peakness of the distribution decreases. Figure 4 shows the effect of C.*

5. *Figure 5 shows that as α decreases, the distribution has fatter tails and increased peakness. Indeed, we can show that the Lévy process corresponding to the MTS distribution has finite activity if $\alpha < 0$ and infinite activity if*

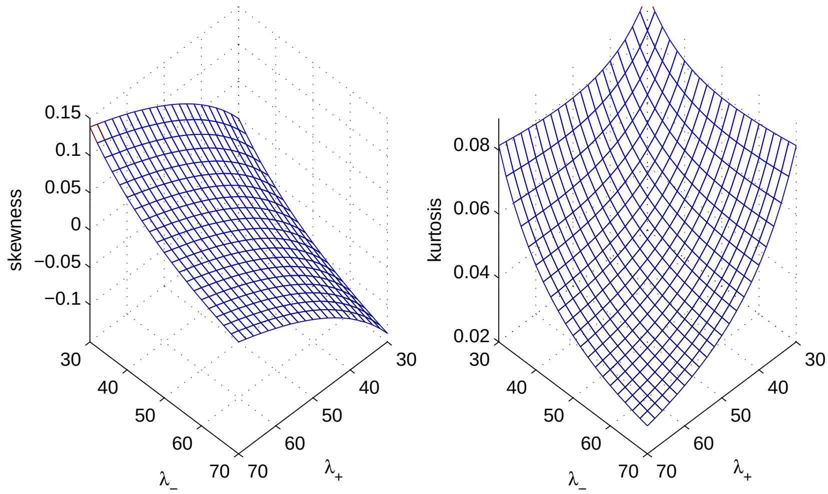

Figure 2: Skewness and Excess Kurtosis of MTS distributions : dependence on λ_+ and λ_-. Parameters : $\alpha = 1.4$, $C = 0.02$, $\mu = 0$, $t = 1$.

$\alpha \geq 0$. *Moreover it has finite variation if* $\alpha < 1$ *and infinite variation if* $\alpha \geq 1$ *(See Proposition 3.5.4 of [10]).*

If we put

$$C = 2^{\frac{\alpha+1}{2}} \left(\sqrt{\pi}\Gamma \left(1 - \frac{\alpha}{2} \right) \left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2} \right) \right)^{-1}$$

and

$$\mu = -2^{-\frac{\alpha+1}{2}} C\Gamma \left(\frac{1-\alpha}{2} \right) \left(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1} \right),$$

then $X \sim MTS(\alpha, C, \lambda_+, \lambda_-, \mu)$ has zero mean and unit variance. In this case, we say that the random variable X has the *standard MTS distribution*, and denote $X \sim stdMTS(\alpha, \lambda_+, \lambda_-)$.

2.4 Measure Change On Modified Tempered Stable Distributions

To apply the MTS distributions to no-arbitrage option pricing, we would need to determine an equivalent martingale measure (EMM). In this section,

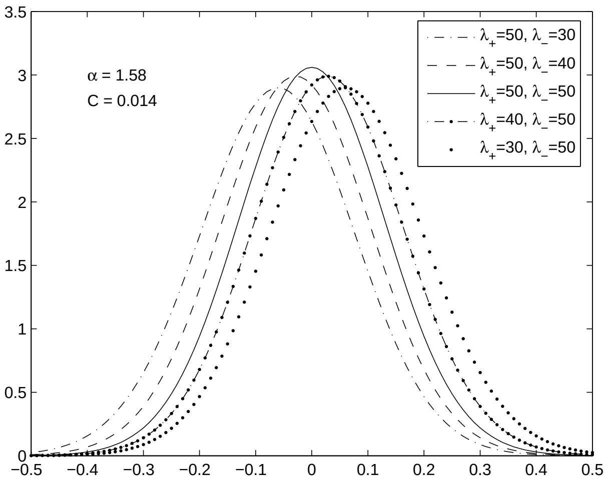

Figure 3: Probability density of the MTS distributions: dependence on λ_+ and λ_-. Parameters : $\lambda_+ = 50$, $\lambda_- \in \{30, 40, 50, 60, 70\}$, $\alpha = 1.58$, $C = 0.02$, $\mu = 0$. figSkewnessOfMTL

we review a general result of equivalence of measures presented by Sato [19] and apply it to the MTS distribution. The following theorem is a particular case of Theorem 33.1 in [19].

Theorem 2.12. *Let $(X, \mathbf{P})$ and $(X, \mathbf{Q})$ be two infinitely divisible random variables on $\mathbb{R}$ with Lévy triplet (σ^2, ν, γ) and $(\tilde{\sigma}^2, \tilde{\nu}, \tilde{\gamma})$ respectively. Then $\mathbf{P}$ and $\mathbf{Q}$ are equivalent if, and only if, the Lévy triplet satisfies*

$$\sigma^2 = \tilde{\sigma}^2, \tag{2.12}$$

$$\int_{-\infty}^{\infty} (e^{\psi(x)/2} - 1)^2 \nu(dx) < \infty, \tag{2.13}$$

where $\psi(x) = \log\left(\frac{\tilde{\nu}(dx)}{\nu(dx)}\right)$. If $\sigma^2 = 0$ then

$$\tilde{\gamma} - \gamma = \int_{|x|\leq 1} x(\tilde{\nu} - \nu)(dx). \tag{2.14}$$

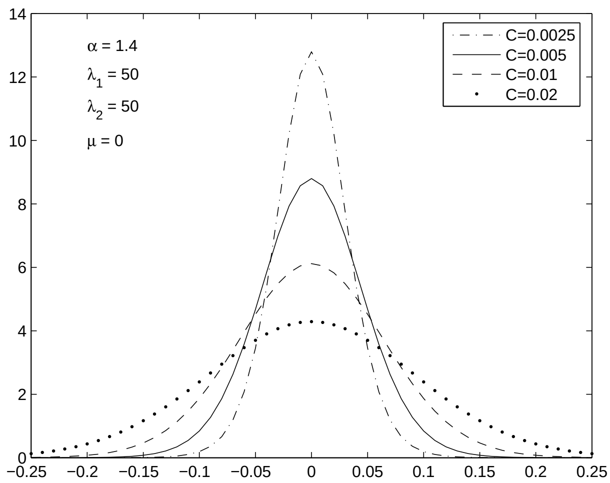

Figure 4: Probability density of the MTS distributions: dependence on C. Parameters : $C \in \{0.0025, 0.005, 0.01, 0.02\}$, $\alpha = 1.4$, $\lambda_+ = 50$, $\lambda_- = 50$, $\mu = 0$.

Since MTS distributions are infinitely divisible, we can apply Theorem 2.12 to obtain the change of measure.

Proposition 2.13. *Let $(X, \mathbf{P})$ and $(X, \mathbf{Q})$ be two MTS distributed random variables on $\mathbb{R}$ with parameters $(\alpha, C, \lambda_+, \lambda_-, \mu)$ and $(\tilde{\alpha}, \tilde{C}, \tilde{\lambda}_+, \tilde{\lambda}_-, \tilde{\mu})$, respectively. Then $\mathbf{P}$ and $\mathbf{Q}$ are equivalent if, and only if, $C = \tilde{C}$, $\alpha = \tilde{\alpha}$ and $\mu = \tilde{\mu}$.*

3 The MTS-GARCH Option Pricing Model

The MTS-GARCH stock price model is defined over a filtered probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\in\mathbb{N}}, \mathbb{P})$ which is constructed as follows: Consider a sequence $(\varepsilon_t)_{t\in\mathbb{N}}$ of iid real random variables on a sequence of probability spaces $(\Omega_t, \mathbf{P}_t)_{t\in\mathbb{N}}$, such that $\varepsilon_t \sim stdMTS(\alpha, \lambda_+, \lambda_-)$ on the space $(\Omega_t, \mathbf{P}_t)$. Next, we define $\Omega := \prod_{t\in\mathbb{N}} \Omega_t$, $\mathfrak{F}_t := \otimes_{k=1}^t \sigma(\varepsilon_k) \otimes \mathfrak{F}_0 \otimes \mathfrak{F}_0 \cdots$, $\mathfrak{F} := \sigma\left(\cup_{t\in\mathbb{N}} \mathfrak{F}_t\right)$, and $\mathbb{P} := \otimes_{t\in\mathbb{N}} \mathbf{P}_t$, where $\mathfrak{F}_0 = \{\emptyset, \Omega\}$ and $\sigma(\varepsilon_k)$ means the σ-algebra generated by ε_k on Ω_k.

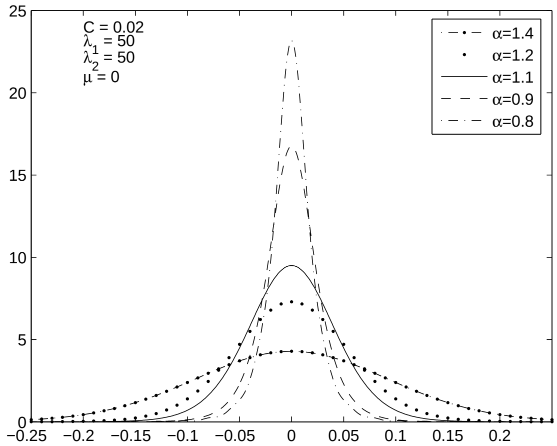

Figure 5: Fat Tail Probability density of the MTS distributions: dependence on α. Parameters : $\alpha \in \{0.8, 0.9, 1.1, 1.2, 1.4\}$, $C = 0.02$, $\lambda_+ = 50$, $\lambda_- = 50$, $\mu = 0$.

We propose the following model for the stock price dynamics:

$$\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t + \lambda_t \sigma_t - g(\sigma_t; \alpha, \lambda_+, \lambda_-) + \sigma_t \varepsilon_t, \quad t \in \mathbb{N}, \tag{3.1}$$

where S_t denotes the price of the underlying asset at time t, r_t and d_t denote the risk free rate and dividend rate for the period $[t-1, t]$, and λ_t is a $\mathfrak{F}_{t-1}$ measurable random variable. S_0 is the present observed price. The function $g(x; \alpha, \lambda_+, \lambda_-)$ is the characteristic exponent of the Laplace transform for the distribution $stdMTS(\alpha, \lambda_+, \lambda_-)$, i.e. $g(x; \alpha, \lambda_+, \lambda_-) = \log(E_{\mathbf{P}_t}[\exp(x\varepsilon_t)])$. The function $g(x; \alpha, \lambda_+, \lambda_-)$ is defined if $x \in (-\lambda_-, \lambda_+)$ and its value can be obtained from (2.8) if $|x| < \lambda_+ \wedge \lambda_-$, and by numerical calculation if $x \in \{x \in (-\lambda_-, \lambda_+) \mid |x| \geq \lambda_+ \wedge \lambda_-\}$. The one period ahead forecast variance σ_t^2 at time $t-1$ follows a GARCH(1,1) process with a restriction $0 < \sigma_t < \lambda_+$, i.e.

$$\sigma_t^2 = (\alpha_0 + \alpha_1 \sigma_{t-1}^2 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2) \wedge \rho, \quad t \in \mathbb{N}, \quad \varepsilon_0 = 0, \tag{3.2}$$

where the coefficients α_0, α_1, and β_1 are non-negative, $\alpha_1 + \beta_1 < 1$, $\alpha_0 > 0$, and $0 < \rho < \lambda_+^2$. Clearly σ_t is $\mathfrak{F}_{t-1}$-measurable and hence the process $(\sigma_t)_{t\in\mathbb{N}}$ is predictable. Moreover, the conditional expectation $E[\hat{S}_t/\hat{S}_{t-1}|\mathfrak{F}_{t-1}]$ equals $\exp(r_t + \lambda_t\sigma_t)$ where $\hat{S}_t = S_t \exp\left(\sum_{k=1}^t d_k\right)$ is the stock price considering reinvestment of the dividends, thus λ_t can be interpreted as the market price of risk.

Remark 3.1. *If ε_t equals the standard normal distributed random variable for all $t \in \mathbb{N}$, g is to be the Laplace transform of ε_t and we ignore the restriction $\sigma_t < \lambda_+$, then the model becomes 'the normal GARCH model' introduced by Duan [6].*

Proposition 3.2. *Let $t \in \mathbb{N}$ be fixed and $\varepsilon_t \sim stdMTS(\alpha, \lambda_+, \lambda_-)$ under $\mathbf{P}_t$. Suppose positive real numbers $\tilde{\lambda}_+$ and $\tilde{\lambda}_-$ satisfy the equation*

$$\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2} = \tilde{\lambda}_+^{\alpha-2} + \tilde{\lambda}_-^{\alpha-2}. \tag{3.3}$$

Let

$$k = 2^{-\frac{\alpha+1}{2}} C\Gamma\left(\frac{1-\alpha}{2}\right)\left(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1} - \tilde{\lambda}_+^{\alpha-1} + \tilde{\lambda}_-^{\alpha-1}\right), \tag{3.4}$$

where

$$C = 2^{\frac{\alpha+1}{2}}\left(\sqrt{\pi}\Gamma\left(1-\frac{\alpha}{2}\right)(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2})\right)^{-1}.$$

Then, there is a probability measure $\mathbf{Q}_t$ equivalent to $\mathbf{P}_t$, such that $(\varepsilon_t + k) \sim stdMTS(\alpha, \tilde{\lambda}_+, \tilde{\lambda}_-)$.

Assumption (A) (i) There exist $\tilde{\lambda}_+$ and $\tilde{\lambda}_-$ satisfying equations (3.3) and $\tilde{\lambda}_+ \geq \lambda_+$. (ii) The market price of risk λ_t is given by $\lambda_t = k - (g(\sigma_t; \alpha, \tilde{\lambda}_+, \tilde{\lambda}_-) - g(\sigma_t; \alpha, \lambda_+, \lambda_-))/\sigma_t$, for each $0 \leq t \leq T$, where k is defined as (3.4).

Under Assumption (A), let $\mathbf{Q}_t$ be the measure described in Proposition 3.2.

Definition 3.3. *Let $T \in \mathbb{N}$ be the time horizon. Define a new measure $\mathbb{Q}$ on $\mathfrak{F}_T$ equivalent to measure $\mathbb{P}$, with a Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$ where* *the density process* $(Z_t)_{0\le t\le T}$ *is defined according to*

$$
\begin{aligned}
Z_0 &\equiv 1,\\
Z_t &:= \frac{d(\mathbf{P}_1\otimes\cdots\otimes\mathbf{P}_{t-1}\otimes\mathbf{Q}_t\otimes\mathbf{P}_{t+1}\otimes\cdots\otimes\mathbf{P}_T)}{d\mathbb{P}} Z_{t-1}, \quad t=1,2,\cdots,T.
\end{aligned}
$$

Lemma 3.4. *The measure* $\mathbb{Q}$ *satisfies the following requirements:*

(a) *The discount asset price process* $(e^{-r_t}\hat{S}_t)_{1\le t\le T}$ *is a* $\mathbb{Q}$*-martingale w.r.t. the filtration* $(\mathfrak{F}_t)_{1\le t\le T}$*.*

(b) *We have*

$$
\mathrm{Var}_{\mathbb{Q}}\left(\log\left(\frac{S_t}{S_{t-1}}\right)\Big|\mathfrak{F}_{t-1}\right) \overset{\text{a.s.}}{=} \mathrm{Var}_{\mathbb{P}}\left(\log\left(\frac{S_t}{S_{t-1}}\right)\Big|\mathfrak{F}_{t-1}\right), \quad 1\le t\le T
$$

(c) *The stock price dynamics under* $\mathbb{Q}$ *can be written as*

$$
\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t - g(\sigma_t;\alpha,\tilde{\lambda}_+,\tilde{\lambda}_-) + \sigma_t\xi_t, \quad 1\le t\le T
$$

where $(\xi_t)_{1\le t\le T}$ *is a sequence of real random variables on* Ω_t *satisfying* $\xi_t \sim stdMTS(\alpha,\tilde{\lambda}_+,\tilde{\lambda}_-)$ *under* $\mathbf{Q}_t$ *for* $1\le t\le T$*. The variance process under* $\mathbb{Q}$ *has the form*

$$
\sigma_t^2 = (\alpha_0 + \alpha_1\sigma_{t-1}^2(\xi_{t-1}-k)^2 + \beta_1\sigma_{t-1}^2)\wedge(\lambda_+^2(1-\epsilon)), \ t\in\mathbb{N}, \ \xi_0 = 0.
$$

The stock price dynamics under $\mathbb{Q}$ which is stated in Lemma 3.4 (c) is called *the MTS-GARCH risk neutral price process.* The arbitrage free price of a call option with strike price K and maturity T is given by

$$
C_t = \exp\left(-\sum_{k=t+1}^{T} r_k\right) E_{\mathbb{Q}}[(S_T-K)^+|\mathfrak{F}_t] \tag{3.5}
$$

where the stock price S_T at time T is given by

$$
S_T = S_t\exp\left(\sum_{k=t+1}^{T}\Big((r_k-d_k) - g(\sigma_t;\alpha,\tilde{\lambda}_+,\tilde{\lambda}_-) + \sigma_k\xi_k\Big)\right).
$$

3.1 Estimation of the Parameters for the GARCH models

In this section, we report on the maximum likelihood estimation (MLE) of both normal-GARCH and MTS-GARCH models. In our empirical study, we use a data set including the S&P 500 index (SPX), International Business Machines (IBM), Johnson and Johnson (JNJ), and 3M (MMM) from April 1, 1996 to March 31, 2006. Data are supplied by Yahoo! Finance. For the daily risk-free rate, we pick the yields of the 3-month T-bills and change them to the continuous compound rate. To simplify the estimation, we impose a constant market price of risk λ. For IBM, JNJ, and MMM, we use the adjusted-closing prices to estimate the market parameters with the MLE. The adjusted-closing prices adjust for all applicable stock splits and stock dividend distributions.

Our estimation procedure is as follows : First, we estimate the parameters α_0, α_1, β_1, and the constant market price of risk λ from the normal-GARCH model. Second, we fix α_0, α_1, β_1, and λ and then estimate α, λ_+, and λ_- from the MTS-GARCH model. Here we assume that $\sigma_0^2 = \alpha_0/(1-\alpha_1-\beta_1)$ and $\rho = \max\{\sigma_t^2 : t \text{ is the observed date.}\}$. We list the estimated GARCH parameters and the parameters for the standard MTS distribution in Table 1.

For the assessment of the goodness-of-fit, we use the Kolmogrov-Smirnov (KS) test. Moreover, we calculate the Anderson-Darling (AD) statistic to better evaluate the tail fit. The KS statistic is defined as

$$KS = \sup_{x_i} |F(x_i) - \hat{F}(x_i)|,$$

and the AD statistic is defined as

$$AD = \sup_{x_i} \frac{|F(x_i) - \hat{F}(x_i)|}{\sqrt{F(x_i)(1-F(x_i))}},$$

where F is the cumulative distribution function and $\hat{F}$ is the empirical cumulative distribution function for a given observation $\{x_i\}$. Table 2 provides the KS statistic and their p-values. The p-values of the KS statistic are calculated

Table 1: Estimated parameters

	GARCH Parameters				Standard MTS		
	β_1	α_1	α_0	λ	α	λ_+	λ_-
SPX	0.9138	0.0767	1.2762e-6	0.0653	1.5479	2.0152	1.0091
IBM	0.9067	0.0904	3.5746e-6	0.0621	1.6705	0.3623	0.4803
JNJ	0.9179	0.0756	2.1964e-5	0.0523	1.4832	0.8032	1.0797
MMM	0.8496	0.1042	1.3184e-6	0.0524	1.4768	0.6163	0.8969

Table 2: Statistic of the goodness of fit tests

	Standard Normal		Standard MTS	
	KS(p-value)	AD	KS(p-value)	AD
SPX	0.0307 (0.0180)	435.19	0.0273 (0.0482)	0.6982
IBM	0.0539 (0.0000)	33665	0.0245 (0.0985)	0.4716
JNJ	0.0395 (0.0008)	3656.0	0.0194 (0.3058)	1.2341
MMM	0.0473 (0.0000)	59987	0.0188 (0.3423)	1.2136

using the calculator designed by Marsaglia et al.[14]. According to this table, p-values of the MTS-GARCH model are larger than those of the normal-GARCH model. Moreover, we can see that the values of the AD statistic for the standard MTS case are significantly smaller than that of the standard normal case. That means the MTS-GARCH model explains the extreme event of the real innovation process better than the normal-GARCH model does. We give an example of QQ-plots for the IBM in Figure 6. The empirical density more or less deviates from the normal distribution and this deviation almost disappears when we use the MTS distributed innovation process.

3.2 Implied Volatility for the GARCH Option Price models

In this section, we discuss the property of the implied volatility for the MTS-GARCH option price model. To determine the risk-neutral parameters, it is necessary to find $\tilde{\lambda}_+$, $\tilde{\lambda}_-$, and k satisfying (A.1) and (A.2) which is impossible if the market price of risk λ_t is a constant. For this reason, the market parameters

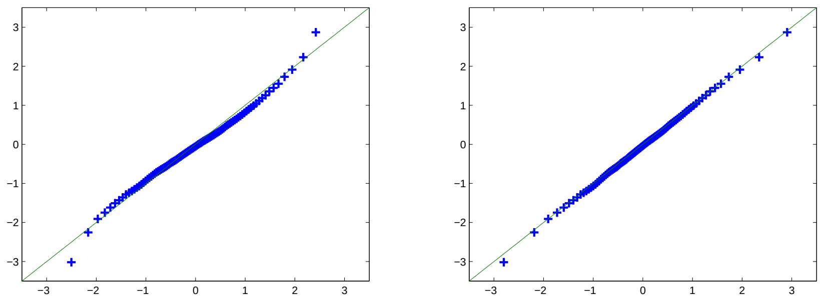

Figure 6: QQ-plots of the MLE fit for the residual distribution of IBM return process. The left graph is the QQ-plot of the standard normal and empirical distribution of the innovation processes, and the right graph is the QQ-plot of the standard MTS and empirical distribution of the innovation processes.

λ_+, λ_-, α, and the risk-neutral parameters $\tilde{\lambda}_+$, $\tilde{\lambda}_-$, k have to be estimated simultaneously, in order to obtain the non-constant market price of risk λ_t. Instead of estimating risk-neutral parameters, we provide an example of the risk-neutral parameters in this paper, and give the implied volatility curve for the call option price given by (3.5).

Since we do not have an efficient analytical form of the option price (3.5), the call option prices are determined by Monte Carlo simulation with 50,000 sample paths. We let $\beta_1 = 0.90$, $\alpha_1 = 0.09$, and $\alpha_0 = 3.5\text{E}-5$. Since the constant market price of risk λ for the normal-GARCH option pricing model plays the same role as the parameter k does for the MTS-GARCH option pricing model, we let $\lambda = k = 0.05$. The daily risk-free rate of return is assumed to be constant $r_t \equiv r = 1.6\text{E}-4$. The risk-neutral parameters of the standard MTS innovation are $\alpha = 1.60$, $\tilde{\lambda}_1 = 0.30$, and $\tilde{\lambda}_2 = 0.10$ and we assume that $\rho = (0.09)^2 = 0.0081$ and $\sigma_0 = 0.0075$. Figure 7 shows the calculated implied volatilities of the MTS-GARCH and the normal-GARCH model prices of call options for given parameters. The curves indicate that the volatility curve for the MTS-GARCH model has larger convexity than the curve of the normal-

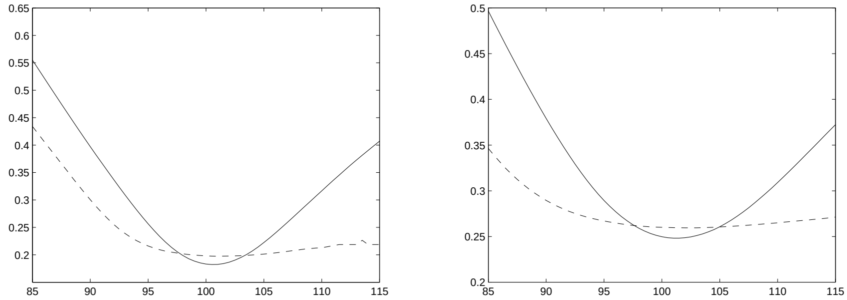

Figure 7: The left picture shows volatility curves for 7 days of the time to maturity, and the right picture shows the curves for 14 days of the time to maturity. The dashed line and solid line indicate the implied volatility curve of the normal-GARCH model and of the MTS-GARCH model, respectively. The x-axis is the strike price and the y-axis is the implied volatility. We assume that $S_0 = 100$.

GARCH model. The skewness of the curve for the MTS-GARCH model is also larger than that of the normal-GARCH model. In the end, we can obtain a more flexible implied volatility curve using the parameters (α, $\tilde{\lambda}_1$, $\tilde{\lambda}_2$) of the MTS innovation distribution.

4 Conclusion

This paper introduces an alternative class of tempered stable distributions which we call the Modified Tempered Stable distribution. It has similar properties as the TS distribution, but it is not fully included in the generalized class of the tempered stable distributions by Rosiński. It can properly describe skewness and kurtosis of asset returns. Next, we introduced an enhanced GARCH-model, namely the MTS-GARCH model, by applying MTS innovations to the classical GARCH model. As a result, the MTS-GARCH time series model for stock returns explains the volatility clustering phenomenon, the leverage effect, and both conditional skewness and leptokurtosis. The risk neutral measure is obtained by applying a change of measure to the MTS distribution.

We obtained encouraging results from the empirical study. The empirical analysis on the S&P 500 index and three different stocks (IBM, JNJ, and MMM) shows that the values of the goodness-of-fit statistics decrease under the GARCH model with MTS innovations. By modeling the innovations with the MTS law, we improved goodness-of-fit statistics for the GARCH model on the S&P 500 index and the data of four different stock prices. Furthermore, the Kolmogrov-Smirnov p-values for the MTS-GARCH model are larger than those for the normal-GARCH model and the Anderson-Darling statistic of the MTS-GARCH model is significantly smaller than that of the normal-GARCH model. In the risk-neutral return process, the MTS-GARCH option pricing model offers a more flexible implied volatility curve than the normal-GARCH model. The skewness and fat tails of the MTS innovations seem to generate the difference. Consequently, the MTS-GARCH model can be a more realistic model than the normal-GARCH model.

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A Appendix

A.1 Special Functions

The *modified Bessel function of the second kind* (See [1, p290]) is defined as

$$K_p(x) = \frac{\pi}{2\sin p\pi}\left(\sum_{k=0}^{\infty}\frac{(x/2)^{2k-p}}{k!\Gamma(k-p+1)} - \sum_{k=0}^{\infty}\frac{(x/2)^{2k+p}}{k!\Gamma(k+p+1)}\right). \tag{A.1}$$

Its asymptotic behavior can be described as follows

$$K_p(x) \sim e^{-x}\sqrt{\frac{\pi}{2x}}, \quad p \geq 0, \quad x \to \infty \tag{A.2}$$

and

$$K_p(x) \sim \frac{\Gamma(p)}{2}\left(\frac{2}{x}\right)^p, \quad p > 0, \quad x \to 0^+. \tag{A.3}$$

The integral representation of $K_p(x)$ is given by

$$K_p(x) = \frac{1}{2}\left(\frac{x}{2}\right)^p \int_0^\infty e^{-t-\frac{x^2}{4t}} t^{-p-1}\, dt$$

and recurrence formula is given by

$$\frac{d}{dx}(x^p K_p(x)) = -x^p K_{p-1}(x). \tag{A.4}$$

The following lemma is useful.

Lemma A.1. *If $\mu - p > -1$ and $a > 0$ then*

$$\int_0^\infty x^\mu K_p(ax)dx = \frac{2^{\mu-1}}{a^{\mu+1}}\Gamma\left(\frac{1+\mu+p}{2}\right)\Gamma\left(\frac{1+\mu-p}{2}\right).$$

Proof. See [1, p299]. □

Now define the hypergeometric function. Before defining it, let us introduce a useful notation

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1), \quad n = 1, 2, 3, \cdots, \quad a \in \mathbb{R} \tag{A.5}$$

called the *Pochhammer symbol* (See [1, p358]). This symbol can also be defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n = 0, 1, 2, 3, \cdots. \tag{A.6}$$

The function

$$F(a, b; c; x) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n}\frac{x^n}{n!}, \quad |x| < 1 \tag{A.7}$$

is called the *hypergeometric function* (See [1, p361]).

Lemma A.2. *For $k = 1, 2, 3 \cdots$,*

$$\frac{d^k}{dx^k} F(a, b; c; x) = \frac{(a)_k (b)_k}{(c)_k} F(a + k, b + k; c + k; x). \tag{A.8}$$

Proof. See [1, p367]. $\square$

A.2 Proofs of Proposition 2.5

Proof. It suffices to show that

$$\int_0^\infty (x^2 \wedge 1) \frac{K_{\frac{\alpha+1}{2}}(\lambda x)}{x^{\frac{\alpha+1}{2}}} dx < \infty.$$

We first note that

$$\int_0^\infty x^2 \exp\left(-\frac{(\lambda x)^2}{4t}\right) dx = \frac{4t^{\frac{3}{2}}}{\lambda^3} \int_0^\infty y^{\frac{1}{2}} e^{-y} dy = \frac{4}{\lambda^3} t^{\frac{3}{2}} \Gamma\left(\frac{3}{2}\right) = \frac{2\sqrt{\pi}}{\lambda^3} t^{\frac{3}{2}}.$$

Hence we have

$$\begin{aligned}
\int_0^\infty x^2 \frac{K_{\frac{\alpha+1}{2}}(\lambda x)}{x^{\frac{\alpha+1}{2}}} dx &= \frac{1}{2} \left(\frac{\lambda}{2}\right)^{\frac{\alpha+1}{2}} \int_0^\infty \int_0^\infty x^2 \exp\left(-\frac{(\lambda x)^2}{4t}\right) dx\; e^{-t} t^{-\left(\frac{\alpha+3}{2}\right)} dt \\
&= \frac{1}{2} \left(\frac{\lambda}{2}\right)^{\frac{\alpha+1}{2}} \frac{2\sqrt{\pi}}{\lambda^3} \int_0^\infty e^{-t} t^{-\frac{\alpha}{2}} dt \\
&= \frac{\lambda^{\frac{\alpha-5}{2}} \sqrt{\pi}}{2^{\frac{\alpha+1}{2}}} \Gamma\left(1 - \frac{\alpha}{2}\right).
\end{aligned}$$

Therefore, we have

$$\int_0^\infty (x^2 \wedge 1) \frac{K_{\frac{\alpha+1}{2}}(\lambda x)}{x^{\frac{\alpha+1}{2}}} dx \le \int_0^\infty x^2 \frac{K_{\frac{\alpha+1}{2}}(\lambda x)}{x^{\frac{\alpha+1}{2}}} dx < \infty.$$

$\square$

A.3 Proofs of Theorem 2.8 and Proposition 2.10

Lemma A.3. *Let $\lambda > 0$. Then*

$$\lambda^{\frac{\alpha+1}{2}} \int_0^1 \frac{K_{\frac{\alpha+1}{2}}(\lambda x)}{x^{\frac{\alpha-1}{2}}} dx = \begin{cases} \frac{\lambda^{\alpha-1}}{2^{\frac{\alpha+1}{2}}} \Gamma\left(\frac{1-\alpha}{2}\right) - \lambda^{\frac{\alpha-1}{2}} K_{\frac{\alpha-1}{2}}(\lambda) & \text{if } \alpha < 1 \\ \infty & \text{if } \alpha \ge 1 \end{cases}.$$

Proof. See Lemma 3.3.1 in [10]. $\square$

Lemma A.4. *Let $u^2 < \lambda^2$.*

1. If $\alpha < 1$, then

$$\text{(A.9)} \qquad \sum_{n=1}^{\infty} \frac{(iu)^n}{n!} \lambda^{\frac{\alpha}{2}+\frac{1}{2}} \int_0^{\infty} x^{n-\frac{\alpha}{2}-\frac{1}{2}} K_{\frac{\alpha+1}{2}}(\lambda x) dx$$
$$= \begin{cases} 2^{-\frac{\alpha}{2}-\frac{3}{2}} \sqrt{\pi} \Gamma(-\frac{\alpha}{2})((\lambda^2+u^2)^{\frac{\alpha}{2}} - \lambda^{\alpha}) & \\ +iu 2^{-\frac{\alpha}{2}-\frac{1}{2}} \lambda^{\alpha-1} \Gamma(\frac{1}{2}-\frac{\alpha}{2}) F\left(1, \frac{1}{2}-\frac{\alpha}{2}; \frac{3}{2}; -\frac{u^2}{\lambda^2}\right) & \text{if } \alpha \neq 0 \\ \sqrt{\pi} 2^{-\frac{3}{2}} \log\left(\frac{\lambda^2}{\lambda^2+u^2}\right) & \\ +iu 2^{-\frac{1}{2}} \lambda^{-1} \Gamma(\frac{1}{2}) F\left(1, \frac{1}{2}; \frac{3}{2}; -\frac{u^2}{\lambda^2}\right) & \text{if } \alpha = 0 \end{cases}.$$

2. If $\alpha \in (1,2)$, then

$$\text{(A.10)} \qquad \sum_{n=2}^{\infty} \frac{(iu)^n}{n!} \lambda^{\frac{\alpha}{2}+\frac{1}{2}} \int_0^{\infty} x^{n-\frac{\alpha}{2}-\frac{1}{2}} K_{\frac{\alpha+1}{2}}(\lambda x) dx$$
$$= \frac{\sqrt{\pi}}{2^{\frac{\alpha}{2}+\frac{3}{2}}} \Gamma(-\frac{\alpha}{2})((\lambda^2+u^2)^{\frac{\alpha}{2}} - \lambda^{\alpha})$$
$$+ \frac{iu\lambda^{\alpha-1}\Gamma(\frac{1}{2}-\frac{\alpha}{2})}{2^{\frac{\alpha}{2}+\frac{1}{2}}} \left(F\left(1, \frac{1}{2}-\frac{\alpha}{2}; \frac{3}{2}; -\frac{u^2}{\lambda^2}\right) - 1 \right).$$

Proof. See Lemma 3.3.2 in [10]. □

Proof of Theorem 2.8. Let

$$H(\alpha, \lambda, u) = \int_0^{\infty} (e^{iux} - 1 - iux 1_{|x|\le 1}) \lambda^{\frac{\alpha}{2}+\frac{1}{2}} \frac{K_{\frac{\alpha+1}{2}}(\lambda x)}{x^{\frac{\alpha}{2}+\frac{1}{2}}} dx,$$

where $\lambda > 0$ and $|iu| < \lambda$. Let $\alpha < 1$. Then, we have

$$H(\alpha, \lambda, u) = \lambda^{\frac{\alpha}{2}+\frac{1}{2}} \int_0^{\infty} (e^{iux} - 1) \frac{K_{\frac{\alpha+1}{2}}(\lambda x)}{x^{\frac{\alpha}{2}+\frac{1}{2}}} dx - iu\lambda^{\frac{\alpha}{2}+\frac{1}{2}} \int_0^1 \frac{K_{\frac{\alpha+1}{2}}(\lambda x)}{x^{\frac{\alpha}{2}-\frac{1}{2}}} dx.$$

By Lemma A.3 and the series expansion of the exponential function, we have

$$H(\alpha, \lambda, u) = \sum_{n=1}^{\infty} \frac{(iu)^n}{n!} \lambda^{\frac{\alpha}{2}+\frac{1}{2}} \int_0^{\infty} x^{n-\frac{\alpha}{2}-\frac{1}{2}} K_{\frac{\alpha+1}{2}}(\lambda x) dx$$
$$- iu \left(\frac{\lambda^{\alpha-1}}{2^{\frac{\alpha}{2}+\frac{1}{2}}} \Gamma\left(\frac{1}{2}-\frac{\alpha}{2}\right) - \lambda^{\frac{\alpha}{2}-\frac{1}{2}} K_{\frac{\alpha-1}{2}}(\lambda) \right).$$

By Lemma A.4, we obtain that

$$
\begin{aligned}
H(\alpha,\lambda,u) &= \sqrt{\pi}2^{-\frac{\alpha}{2}-\frac{3}{2}}\Gamma\left(-\frac{\alpha}{2}\right)((\lambda^2+u^2)^{\frac{\alpha}{2}}-\lambda^\alpha)1_{\alpha\neq 0}\\
&+\sqrt{\pi}2^{-\frac{3}{2}}\log\left(\frac{\lambda^2}{\lambda^2+u^2}\right)1_{\alpha=0}+\frac{iu\lambda^{\alpha-1}\Gamma(\frac{1}{2}-\frac{\alpha}{2})}{2^{\frac{\alpha}{2}+\frac{1}{2}}}F\left(1,\frac{1}{2}-\frac{\alpha}{2};\frac{3}{2};-\frac{u^2}{\lambda^2}\right)\\
&-iu\left(\frac{\lambda^{\alpha-1}}{2^{\frac{\alpha}{2}+\frac{1}{2}}}\Gamma\left(\frac{1}{2}-\frac{\alpha}{2}\right)-\lambda^{\frac{\alpha}{2}-\frac{1}{2}}K_{\frac{\alpha-1}{2}}(\lambda)\right).
\end{aligned}
$$

Let $\alpha\in(1,2)$. Then, we have

$$
H(\alpha,\lambda,u)=\lambda^{\frac{\alpha}{2}+\frac{1}{2}}\int_0^\infty(e^{iux}-1-iux)\frac{K_{\frac{\alpha+1}{2}}(\lambda x)}{x^{\frac{\alpha}{2}+\frac{1}{2}}}dx+iu\lambda^{\frac{\alpha}{2}+\frac{1}{2}}\int_1^\infty\frac{K_{\frac{\alpha+1}{2}}(\lambda x)}{x^{\frac{\alpha}{2}-\frac{1}{2}}}dx.
$$

Since we can show that $\lambda^{\frac{\alpha}{2}+\frac{1}{2}}\int_1^\infty x^{-\frac{\alpha}{2}+\frac{1}{2}}K_{\frac{\alpha+1}{2}}(\lambda x)dx=\lambda^{\frac{\alpha}{2}-\frac{1}{2}}K_{\frac{\alpha-1}{2}}(\lambda)$ for $\alpha\in(1,2)$, we have

$$
H(\alpha,\lambda,u)=\lambda^{\frac{\alpha}{2}+\frac{1}{2}}\int_0^\infty(e^{iux}-1-iux)\frac{K_{\frac{\alpha+1}{2}}(\lambda x)}{x^{\frac{\alpha}{2}+\frac{1}{2}}}dx+iu\lambda^{\frac{\alpha}{2}-\frac{1}{2}}K_{\frac{\alpha-1}{2}}(\lambda).
$$

By the series expansion of the exponential function, we obtain

$$
H(\alpha,\lambda,u)=\sum_{n=2}^\infty\frac{(iu)^n}{n!}\lambda^{\frac{\alpha}{2}+\frac{1}{2}}\int_0^\infty x^{n-\frac{\alpha}{2}-\frac{1}{2}}K_{\frac{\alpha+1}{2}}(\lambda x)dx+iu\lambda^{\frac{\alpha}{2}-\frac{1}{2}}K_{\frac{\alpha-1}{2}}(\lambda).
$$

By Lemma A.4, we have

$$
\begin{aligned}
H(\alpha,\lambda,u) &= \frac{\sqrt{\pi}}{2^{\frac{\alpha}{2}+\frac{3}{2}}}\Gamma\left(-\frac{\alpha}{2}\right)((\lambda^2+u^2)^{\frac{\alpha}{2}}-\lambda^\alpha)\\
&+\frac{iu\lambda^{\alpha-1}\Gamma(\frac{1}{2}-\frac{\alpha}{2})}{2^{\frac{\alpha}{2}+\frac{1}{2}}}F\left(1,\frac{1}{2}-\frac{\alpha}{2};\frac{3}{2};-\frac{u^2}{\lambda^2}\right)-iu\left(\frac{\Gamma(\frac{1}{2}-\frac{\alpha}{2})}{2^{\frac{\alpha}{2}+\frac{1}{2}}}\lambda^{\alpha-1}-\lambda^{\frac{\alpha}{2}-\frac{1}{2}}K_{\frac{\alpha-1}{2}}(\lambda)\right).
\end{aligned}
$$

So, for $\alpha\in(-\infty,1)\cup(1,2)$ and $|iu|<\lambda_+\wedge\lambda_-$, we have

$$
\begin{aligned}
& iu\gamma+\int_{-\infty}^\infty(e^{iux}-1-iux1_{|x|\leq 1})\nu(dx)\\
&= iu\gamma+CH(\alpha,\lambda_+,u)+C\overline{H(\alpha,\lambda_-,u)}\\
&= iu\mu+G_R(u;\alpha,C,\lambda_+,\lambda_-)+G_I(u;\alpha,C,\lambda_+,\lambda_-).
\end{aligned}
$$

By the Lévy-Kintchine formula, we obtain the desired characteristic function in the theorem. The characteristic function $\phi_X(u)$ can be extended via analytic continuation to the region $\{z\in\mathbb{C}\ :\ |\mathrm{Im}(z)|<\lambda_+\wedge\lambda_-\}$. □

Proof of Proposition 2.10. We first note that, if h is an infinitely differentiable function, then we have, for $n \in \mathbb{N}$ and $k \in \mathbb{R}$,

$$\frac{d^{2n+1}}{du^{2n+1}}(uh(ku^2))\Big|_{u=0} = 2^n \cdot 1 \cdot 3 \cdots (2n+1) k^n h^{(n)}(0),$$

and

$$\frac{d^{2n}}{du^{2n}}(uh(ku^2))\Big|_{u=0} = 0.$$

By this note and (A.8) , we obtain that

$$\begin{aligned}
&\frac{d^{2n+1}}{du^{2n+1}}\left(uF\left(1, \frac{1}{2}-\frac{\alpha}{2}; \frac{3}{2}; \frac{u^2}{\lambda^2}\right)\right)\Big|_{u=0} \\
&= 2^n \cdot 1 \cdot 3 \cdots (2n+1)\lambda^{-2n}\frac{(1)_n(\frac{1}{2}-\frac{\alpha}{2})_n}{(\frac{3}{2})_n}F\left(1+n, \frac{1}{2}-\frac{\alpha}{2}+n; \frac{3}{2}+n; 0\right) \\
&= \frac{(2n+1)!(\frac{1}{2}-\frac{\alpha}{2})_n}{(\frac{3}{2})_n\lambda^{2n}} = \left(\frac{2}{\lambda}\right)^{2n} n! \frac{\Gamma\left(n+\frac{1}{2}-\frac{\alpha}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{\alpha}{2}\right)}
\end{aligned}$$

and

$$\frac{d^{2n}}{du^{2n}}\left(uF\left(1, \frac{1}{2}-\frac{\alpha}{2}; \frac{3}{2}; \frac{u^2}{\lambda^2}\right)\right)\Big|_{u=0} = 0.$$

Hence we have, for $m \in \mathbb{N}$,

(A.11)

$$\frac{d^m}{du^m}\left(uF\left(1, \frac{1}{2}-\frac{\alpha}{2}; \frac{3}{2}; \frac{u^2}{\lambda^2}\right)\right)\Big|_{u=0} = \begin{cases} \left(\frac{2}{\lambda}\right)^{m-1}\left(\frac{m-1}{2}\right)!\frac{\Gamma\left(\frac{m}{2}-\frac{\alpha}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{\alpha}{2}\right)} & \text{if } m = 1, 3, 5, \cdots \\ 0 & \text{if } m = 2, 4, 6, \cdots \end{cases}$$

On the other hand, if $\alpha \neq 0$, we have

$$\frac{d^{2n}}{du^{2n}}((\lambda^2-u^2)^{\frac{\alpha}{2}}-\lambda^\alpha)\big|_{u=0} = \frac{(2n)!}{n!}(-\frac{\alpha}{2})_n\lambda^{2(\frac{\alpha}{2}-n)} = \frac{(2n)!}{n!}\frac{\Gamma(n-\frac{\alpha}{2})}{\Gamma(-\frac{\alpha}{2})}\lambda^{2(\frac{\alpha}{2}-n)}$$

and

$$\frac{d^{2n+1}}{du^{2n+1}}((\lambda^2-u^2)^{\frac{\alpha}{2}}-\lambda^\alpha)\big|_{u=0} = 0,$$

so we obtain that

(A.12)

$$\frac{d^m}{du^m}((\lambda^2-u^2)^{\frac{\alpha}{2}}-\lambda^\alpha)\Big|_{u=0} = \begin{cases} 0 & \text{if } m = 1, 3, 5, \cdots \\ \frac{m!}{(\frac{m}{2})!}\frac{\Gamma\left(\frac{m}{2}-\frac{\alpha}{2}\right)}{\Gamma(-\frac{\alpha}{2})}\lambda^{\alpha-m} & \text{if } m = 2, 4, 6, \cdots \end{cases}.$$

For $m \in \mathbb{N}$ and $\alpha \neq 0$, the cumulant $c_m(X)$ is given by

$$
\begin{aligned}
\text{(A.13)} \quad c_m(X) &= \frac{d^m}{du^m}(\log E[e^{uX}])\big|_{u=0} \\
&= \frac{d^m}{du^m}(\mu u) + \frac{\sqrt{\pi} C\Gamma(-\frac{\alpha}{2})}{2^{\frac{\alpha}{2}+\frac{3}{2}}} \sum_{j=1}^{2} \left[\frac{d^m}{du^m}((\lambda_j^2 - u^2)^{\frac{\alpha}{2}} - \lambda_j^{\frac{\alpha}{2}}) \right]_{u=0} \\
&- \frac{C\Gamma\left(\frac{1}{2} - \frac{\alpha}{2}\right)}{2^{\frac{\alpha}{2}+\frac{1}{2}}} \sum_{j=1}^{2} \left[(-1)^j \lambda_j^{\alpha-1} \frac{d^m}{du^m} \left(uF\left(1, \frac{1}{2} - \frac{\alpha}{2}; \frac{3}{2}; \frac{u^2}{\lambda_j^2}\right)\right)\right]_{u=0}.
\end{aligned}
$$

Substituting (A.11) and (A.12) into (A.13), we obtain (2.11).

Moreover, since we have

$$
\frac{d^m}{du^m} \log\left(\frac{\lambda^2}{\lambda^2+u^2}\right)\Big|_{u=0} = \begin{cases} 0 & \text{if } m = 1, 3, 5, \cdots \\ \frac{m!}{(\frac{m}{2})!}\Gamma\left(\frac{m}{2}\right)\lambda^{-m} & \text{if } m = 2, 4, 6, \cdots \end{cases},
$$

we obtain (2.11) by the similar arguments given above. □

A.4 Proofs of Proposition 2.13

Lemma A.5. *Let $\lambda > 0$, $\alpha \in (0, 2)$. Then we have*

$$
\begin{aligned}
\text{(1)} \quad & (\lambda x)^{\frac{\alpha}{2}+\frac{1}{2}} K_{\frac{\alpha+1}{2}}(\lambda x) = 2^{\frac{\alpha}{2}-\frac{1}{2}}\Gamma\left(\frac{\alpha}{2}+\frac{1}{2}\right) \\
&+ \frac{2^{\frac{\alpha}{2}-\frac{1}{2}}\pi}{\cos\left(\frac{\alpha}{2}\pi\right)} \left(\sum_{k=1}^{\infty} \frac{(\lambda x/2)^{2k}}{k!\Gamma\left(k - \left(\frac{\alpha}{2}+\frac{1}{2}\right)+1\right)} - \sum_{k=0}^{\infty} \frac{(\lambda x/2)^{2k+\alpha+1}}{k!\Gamma\left(k+\frac{\alpha}{2}+\frac{1}{2}+1\right)} \right) \\
\text{(2)} \quad & \int_0^1 x^{-\alpha} \left(\sum_{k=1}^{\infty} \frac{(\lambda x/2)^{2k}}{k!\Gamma\left(k - \left(\frac{\alpha}{2}+\frac{1}{2}\right)+1\right)} - \sum_{k=0}^{\infty} \frac{(\lambda x/2)^{2k+\alpha+1}}{k!\Gamma\left(k+\frac{\alpha}{2}+\frac{1}{2}+1\right)} \right) dx \\
&= \frac{-\cos(\frac{\alpha}{2}\pi)}{2^{\frac{\alpha}{2}-\frac{1}{2}}\pi} \lambda^{\frac{\alpha}{2}-\frac{1}{2}} K_{\frac{\alpha-1}{2}}(\lambda) - \frac{1}{2\Gamma\left(\frac{3}{2}-\frac{\alpha}{2}\right)} + \frac{\lambda^{\alpha-1}}{2^{\alpha}\Gamma\left(\frac{\alpha}{2}+\frac{1}{2}\right)}
\end{aligned}
$$

Proof. (1) The series form of the modified Bessel function of the second kind is given by (A.1). Hence we have

$$
\begin{aligned}
&(\lambda x)^{\frac{\alpha}{2}+\frac{1}{2}} K_{\frac{\alpha+1}{2}}(\lambda x) \\
&= (\lambda x)^{\frac{\alpha}{2}+\frac{1}{2}} \frac{\pi}{2\sin\left(\frac{\alpha}{2}+\frac{1}{2}\right)\pi} \left(\sum_{k=0}^{\infty} \frac{(\lambda x/2)^{2k-\frac{\alpha}{2}+\frac{1}{2}}}{k!\Gamma\left(k-\left(\frac{\alpha}{2}+\frac{1}{2}\right)+1\right)} \right. \\
&\qquad\qquad\qquad\qquad \left. - \sum_{k=0}^{\infty} \frac{(\lambda x/2)^{2k+\frac{\alpha}{2}+\frac{1}{2}}}{k!\Gamma\left(k+\left(\frac{\alpha}{2}+\frac{1}{2}\right)+1\right)} \right) \\
&= \frac{2^{\frac{\alpha}{2}-\frac{1}{2}}\pi}{\sin\left(\frac{\alpha}{2}+\frac{1}{2}\right)\pi} \left(\frac{1}{\Gamma\left(1-\frac{\alpha}{2}-\frac{1}{2}\right)} + \sum_{k=1}^{\infty} \frac{(\lambda x/2)^{2k}}{k!\Gamma\left(k-\left(\frac{\alpha}{2}+\frac{1}{2}\right)+1\right)} \right. \\
&\qquad\qquad\qquad\qquad \left. - \sum_{k=0}^{\infty} \frac{(\lambda x/2)^{2k+\alpha+1}}{k!\Gamma\left(k+\left(\frac{\alpha}{2}+\frac{1}{2}\right)+1\right)} \right)
\end{aligned}
$$

Since $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$ and $\sin\left(\frac{\alpha}{2}+\frac{1}{2}\right)\pi = \cos(\frac{\alpha}{2}\pi)$, we obtain the result.

(2) We can calculate the following equation

$$
\begin{aligned}
&\int_0^1 x^{-\alpha} \left(\sum_{k=1}^{\infty} \frac{(\lambda x/2)^{2k}}{k!\Gamma\left(k-\left(\frac{\alpha}{2}+\frac{1}{2}\right)+1\right)} - \sum_{k=0}^{\infty} \frac{(\lambda x/2)^{2k+\alpha+1}}{k!\Gamma\left(k+\frac{\alpha}{2}+\frac{1}{2}+1\right)} \right) dx \\
&= \left(\frac{\lambda^{\frac{\alpha}{2}-\frac{1}{2}}}{2^{\frac{\alpha}{2}+\frac{1}{2}}} \right) \left(\sum_{k=1}^{\infty} \frac{(\lambda/2)^{2k-\left(\frac{\alpha}{2}-\frac{1}{2}\right)}}{k!\Gamma\left(k-\left(\frac{\alpha}{2}-\frac{1}{2}\right)+1\right)} - \sum_{k=1}^{\infty} \frac{(\lambda/2)^{2k+\left(\frac{\alpha}{2}-\frac{1}{2}\right)}}{k!\Gamma\left(k+\left(\frac{\alpha}{2}-\frac{1}{2}\right)+1\right)} \right).
\end{aligned}
$$

By the series form of $K_{\frac{\alpha-1}{2}}(\lambda)$, we can obtain the result. □

Proof of Proposition 2.13. Let (σ,ν,γ) and $(\tilde{\sigma},\tilde{\nu},\tilde{\gamma})$ be Lévy triplets of $X_{\mathbf{P}}$ and $X_{\mathbf{Q}}$, respectively. Since $\sigma=\tilde{\sigma}=0$, (2.12) is satisfied. Let

$$
k(\alpha,\lambda,x) = \frac{\lambda^{\frac{\alpha}{2}+\frac{1}{2}} K_{\frac{\alpha+1}{2}}(\lambda x)}{x^{\frac{\alpha}{2}+\frac{1}{2}}}.
$$

Then the function $\psi(x) = \ln \frac{d\tilde{\nu}(x)}{d\nu(x)}$ is given by

$$
\psi(x) = \ln\left(\frac{\tilde{C}k(\tilde{\alpha},\tilde{\lambda}_+,x)}{Ck(\alpha,\lambda_+,x)} \right) 1_{x>0} + \ln\left(\frac{\tilde{C}k(\tilde{\alpha},\tilde{\lambda}_-,x)}{Ck(\alpha,\lambda_-,x)} \right) 1_{x<0},
$$

and so

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \left(e^{\frac{\psi(x)}{2}}-1\right)^2 \nu(dx) \\
&= \int_0^{\infty} \left(\sqrt{\tilde{C}}k(\tilde{\alpha},\tilde{\lambda}_+,x)^{\frac{1}{2}} - \sqrt{C}k(\alpha,\lambda_+,x)^{\frac{1}{2}} \right)^2 dx \\
&\quad + \int_{-\infty}^{0} \left(\sqrt{\tilde{C}}k(\tilde{\alpha},\tilde{\lambda}_-,x)^{\frac{1}{2}} - \sqrt{C}k(\alpha,\lambda_-,x)^{\frac{1}{2}} \right)^2 dx.
\end{aligned}
$$

If $\alpha < \tilde{\alpha}$, then for $j = 1, 2$, we have

$$\lim_{x\to 0} \frac{\sqrt{\tilde{C}}k(\tilde{\alpha}, \tilde{\lambda}_j, x)^{\frac{1}{2}} - \sqrt{C}k(\alpha, \lambda_j, x)^{\frac{1}{2}}}{x^{-\frac{\tilde{\alpha}}{2}-\frac{1}{2}}} = \sqrt{\tilde{C}2^{\frac{\tilde{\alpha}}{2}-\frac{1}{2}}\Gamma\left(\frac{\tilde{\alpha}}{2}+\frac{1}{2}\right)}.$$

If $\alpha = \tilde{\alpha}$ but $C < \tilde{C}$, then for $j = 1, 2$, we have

$$\lim_{x\to 0} \frac{\sqrt{\tilde{C}}k(\tilde{\alpha}, \tilde{\lambda}_j, x)^{\frac{1}{2}} - \sqrt{C}k(\alpha, \lambda_j, x)^{\frac{1}{2}}}{x^{-\frac{\tilde{\alpha}}{2}-\frac{1}{2}}} = (\sqrt{\tilde{C}} - \sqrt{C})\sqrt{2^{\frac{\tilde{\alpha}}{2}-\frac{1}{2}}\Gamma\left(\frac{\tilde{\alpha}}{2}+\frac{1}{2}\right)}.$$

Hence if $\alpha < \tilde{\alpha}$ or $\alpha = \tilde{\alpha}$ and $C < \tilde{C}$, then $\left(e^{\frac{\psi(x)}{2}} - 1\right)^2$ is equivalent to $x^{-\tilde{\alpha}-1}$ near zero, so it is not integrable.

Suppose $\alpha = \tilde{\alpha}$ and $C = \tilde{C}$. Then we have

$$\psi(x) = \ln\left(\frac{k(\alpha, \tilde{\lambda}_+, x)}{k(\alpha, \lambda_+, x)}\right)1_{x>0} + \ln\left(\frac{k(\alpha, \tilde{\lambda}_-, x)}{k(\alpha, \lambda_-, x)}\right)1_{x<0}.$$

We can show that $\lim_{x\to 0}\psi(x) = 0$ and $\lim_{x\to 0}\psi'(x) = 0$. Hence, there is a θ such that $\psi(x) < \theta|x|$ for $x \in [-1, 1]$. Thus

$$\int_{|x|\le 1}\left(e^{\frac{\psi(x)}{2}} - 1\right)^2\nu(dx) \le \int_{|x|\le 1}\left(e^{\frac{\theta|x|}{2}} - 1\right)^2\nu(dx) < \infty,$$

and

$$\int_1^\infty\left(e^{\frac{\psi(x)}{2}} - 1\right)^2\nu(dx) \le \int_1^\infty\tilde{\nu}(dx) + \int_1^\infty\nu(dx) < \infty,$$

and similarly, we can show that

$$\int_{-\infty}^{-1}\left(e^{\frac{\psi(x)}{2}} - 1\right)^2\nu(dx) < \infty.$$

Therefore, the condition (2.13) holds if, and only if, $\alpha = \tilde{\alpha}$ and $C = \tilde{C}$.

We have, by Lemma A.5 (1),

$$\begin{aligned}
&\int_0^1 x^{-\alpha}\left((\tilde{\lambda}_\pm x)^{\frac{\alpha}{2}+\frac{1}{2}}K_{\frac{\alpha+1}{2}}(\tilde{\lambda}_\pm x) - (\lambda_\pm x)^{\frac{\alpha}{2}+\frac{1}{2}}K_{\frac{\alpha+1}{2}}(\lambda_\pm x)\right)dx \\
&= \frac{2^{\frac{\alpha}{2}-\frac{1}{2}}\pi}{\cos\left(\frac{\alpha}{2}\pi\right)}\int_0^1\left(\sum_{k=1}^\infty\frac{\left(\tilde{\lambda}_\pm x/2\right)^{2k}}{k!\Gamma\left(k-\left(\frac{\alpha}{2}+\frac{1}{2}\right)+1\right)} - \sum_{k=0}^\infty\frac{\left(\tilde{\lambda}_\pm x/2\right)^{2k+\alpha+1}}{k!\Gamma\left(k+\frac{\alpha}{2}+\frac{1}{2}+1\right)}\right)x^{-\alpha}dx \\
&- \frac{2^{\frac{\alpha}{2}-\frac{1}{2}}\pi}{\cos\left(\frac{\alpha}{2}\pi\right)}\int_0^1\left(\sum_{k=1}^\infty\frac{(\lambda_\pm x/2)^{2k}}{k!\Gamma\left(k-\left(\frac{\alpha}{2}+\frac{1}{2}\right)+1\right)} - \sum_{k=0}^\infty\frac{(\lambda_\pm x/2)^{2k+\alpha+1}}{k!\Gamma\left(k+\frac{\alpha}{2}+\frac{1}{2}+1\right)}\right)x^{-\alpha}dx.
\end{aligned}$$

By the Lemma A.5 (2) and the fact that $\frac{\pi}{\cos(\frac{\alpha}{2}\pi)} = \frac{\pi}{\sin(\frac{\alpha}{2}+\frac{1}{2})\pi} = \Gamma\left(\frac{\alpha}{2}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\alpha}{2}\right)$, we obtain

(A.14)

$$\begin{aligned}
&\int_0^1 x^{-\alpha}\left((\tilde{\lambda}_\pm x)^{\frac{\alpha}{2}+\frac{1}{2}} K_{\frac{\alpha+1}{2}}(\tilde{\lambda}_\pm x) - (\lambda_\pm x)^{\frac{\alpha}{2}+\frac{1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_\pm x)\right) dx \\
&= \frac{2^{\frac{\alpha}{2}-\frac{1}{2}}\pi}{\cos\left(\frac{\alpha}{2}\pi\right)}\left[\frac{-\cos(\frac{\alpha}{2}\pi)}{2^{\frac{\alpha}{2}-\frac{1}{2}}\pi}\tilde{\lambda}_\pm^{\frac{\alpha}{2}-\frac{1}{2}} K_{\frac{\alpha-1}{2}}(\tilde{\lambda}_\pm) + \frac{\tilde{\lambda}_\pm^{\alpha-1}}{2^\alpha\Gamma\left(\frac{\alpha}{2}+\frac{1}{2}\right)}\right. \\
&\qquad\qquad \left. + \frac{\cos(\frac{\alpha}{2}\pi)}{2^{\frac{\alpha}{2}-\frac{1}{2}}\pi}\lambda_\pm^{\frac{\alpha}{2}-\frac{1}{2}} K_{\frac{\alpha-1}{2}}(\lambda_\pm) - \frac{\lambda_\pm^{\alpha-1}}{2^\alpha\Gamma\left(\frac{\alpha}{2}+\frac{1}{2}\right)}\right] \\
&= -\left(\tilde{\lambda}_\pm^{\frac{\alpha}{2}-\frac{1}{2}} K_{\frac{\alpha-1}{2}}(\tilde{\lambda}_\pm) - \lambda_\pm^{\frac{\alpha}{2}-\frac{1}{2}} K_{\frac{\alpha-1}{2}}(\lambda_\pm)\right) + \frac{\Gamma\left(\frac{1}{2}-\frac{\alpha}{2}\right)}{2^{\frac{\alpha}{2}+\frac{1}{2}}}(\tilde{\lambda}_\pm^{\alpha-1} - \lambda_\pm^{\alpha-1})
\end{aligned}$$

Providing that the $\alpha = \tilde{\alpha}$ and $C = \tilde{C}$, the condition (2.14) is equal to

$$\begin{aligned}
&\tilde{\mu} + C\left(\frac{\Gamma\left(\frac{1}{2}-\frac{\alpha}{2}\right)}{2^{\frac{\alpha}{2}+\frac{1}{2}}}\left(\tilde{\lambda}_+^{\alpha-1} - \tilde{\lambda}_-^{\alpha-1}\right) - \tilde{\lambda}_+^{\frac{\alpha}{2}-\frac{1}{2}} K_{\frac{\alpha-1}{2}}(\tilde{\lambda}_+) + \tilde{\lambda}_-^{\frac{\alpha}{2}-\frac{1}{2}} K_{\frac{\alpha-1}{2}}(\tilde{\lambda}_-)\right) \\
&\quad - \mu - C\left(\frac{\Gamma\left(\frac{1}{2}-\frac{\alpha}{2}\right)}{2^{\frac{\alpha}{2}+\frac{1}{2}}}\left(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}\right) - \lambda_+^{\frac{\alpha}{2}-\frac{1}{2}} K_{\frac{\alpha-1}{2}}(\lambda_+) + \lambda_-^{\frac{\alpha}{2}-\frac{1}{2}} K_{\frac{\alpha-1}{2}}(\lambda_-)\right) \\
&= C\int_0^1 x^{-\alpha}\left((\tilde{\lambda}_+ x)^{\frac{\alpha}{2}+\frac{1}{2}} K_{\frac{\alpha+1}{2}}(\tilde{\lambda}_+ x) - (\lambda_+ x)^{\frac{\alpha}{2}+\frac{1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_+ x)\right) dx \\
&\quad - C\int_0^1 x^{-\alpha}\left((\tilde{\lambda}_- x)^{\frac{\alpha}{2}+\frac{1}{2}} K_{\frac{\alpha+1}{2}}(\tilde{\lambda}_- x) - (\lambda_- x)^{\frac{\alpha}{2}+\frac{1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_- x)\right) dx.
\end{aligned}$$

Hence, by the equation (A.14), the condition holds if, and only if, $\tilde{\mu} = \mu$. □

A.5 Proofs of Proposition 3.2 and Lemma 3.4

Proof of Proposition 3.2. Let $t \in \mathbb{N}$ be fixed, $\tilde{\lambda}_+$ and $\tilde{\lambda}_-$ satisfy equations (3.3) and $\xi_t = \varepsilon_t + k$ where k is defined as (3.4). Then $\xi_t \sim MTS(\alpha, C, \lambda_+, \lambda_-, \mu + k)$, where

$$C = 2^{\frac{\alpha}{2}+\frac{1}{2}}\left(\sqrt{\pi}\Gamma\left(1-\frac{\alpha}{2}\right)\left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}\right)\right)^{-1}$$

and

$$\mu = -2^{-\frac{\alpha}{2}-\frac{1}{2}} C\Gamma\left(\frac{1}{2}-\frac{\alpha}{2}\right)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}).$$

For any $\lambda_+, \lambda_- > 0$, put

$$\tilde{\mu}_{\tilde{\lambda}_+, \tilde{\lambda}_-} = \mu + k$$

then $\xi_t \sim MTS(\alpha, C, \tilde{\lambda}_+, \tilde{\lambda}_-, \tilde{\mu}_{\tilde{\lambda}_+,\tilde{\lambda}_-})$ under the probability measure $\mathbf{Q}_{\tilde{\lambda}_+,\tilde{\lambda}_-}$ Under $\mathbf{Q}_{\tilde{\lambda}_+,\tilde{\lambda}_-}$, the variance equals

$$\begin{aligned}\mathrm{Var}_{\mathbf{Q}_{\tilde{\lambda}_+,\tilde{\lambda}_-}}(\xi_t) &= \frac{\sqrt{\pi}\Gamma(1-\frac{\alpha}{2})}{2^{\frac{\alpha}{2}+\frac{1}{2}}} C(\tilde{\lambda}_+^{\alpha-2} + \tilde{\lambda}_-^{\alpha-2}) \\ &= \frac{\tilde{\lambda}_+^{\alpha-2} + \tilde{\lambda}_-^{\alpha-2}}{\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}}\end{aligned}$$

and the mean equals

$$\begin{aligned}E_{\mathbf{Q}_{\tilde{\lambda}_+,\tilde{\lambda}_-}}(\xi_t) &= \tilde{\mu}_{\tilde{\lambda}_+,\tilde{\lambda}_-} + 2^{-\frac{\alpha}{2}-\frac{1}{2}} C\Gamma\left(\frac{1}{2}-\frac{\alpha}{2}\right)(\tilde{\lambda}_+^{\alpha-1} - \tilde{\lambda}_-^{\alpha-1}) \\ &= k - 2^{-\frac{\alpha}{2}-\frac{1}{2}} C\Gamma\left(\frac{1}{2}-\frac{\alpha}{2}\right)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1} - \tilde{\lambda}_+^{\alpha-1} + \tilde{\lambda}_-^{\alpha-1})\end{aligned}$$

By (3.3) and (3.4), we have $E_{\mathbf{Q}_{\tilde{\lambda}_+,\tilde{\lambda}_-}}(\xi_t) = 0$ and $\mathrm{Var}_{\mathbf{Q}_{\tilde{\lambda}_+,\tilde{\lambda}_-}}(\xi_t) = 1$. Hence, let $\mathbf{Q}_t = \mathbf{Q}_{\tilde{\lambda}_+,\tilde{\lambda}_-}$. Then $\mathbf{Q}_t$ and $\mathbf{P}_t$ are equivalent and $\xi_t \sim stdMTS(\alpha, \tilde{\lambda}_+, \tilde{\lambda}_-)$. □

Proof of Lemma 3.4. Let $\varepsilon_t \sim stdMTS(\alpha, \lambda_+, \lambda_-)$, $t \in \mathbb{N}$.

(a) Using Definition 3.3, the Laplace transform of the MTS distribution and the measurability of σ_t with respect to $\mathfrak{F}_{t-1}$, we obtain

$$\begin{aligned}E_{\mathbb{Q}}[\hat{S}_t|\mathfrak{F}_{t-1}] &= E_{\mathbb{Q}}[\hat{S}_{t-1}\exp(r_t + \lambda_t\sigma_t - g(\sigma_t;\alpha,\lambda_+,\lambda_-) + \sigma_t\varepsilon_t)|\mathfrak{F}_{t-1}] \\ &= E_{\mathbb{Q}}[\hat{S}_{t-1}\exp(r_t - g(\sigma_t;\alpha,\tilde{\lambda}_+,\tilde{\lambda}_-) + \sigma_t(k+\varepsilon_t))|\mathfrak{F}_{t-1}] \\ &= \hat{S}_{t-1}\exp(r_t - g(\sigma_t;\alpha,\tilde{\lambda}_+,\tilde{\lambda}_-))E_{\mathbb{Q}}[\exp(\sigma_t(\varepsilon_t+k))|\mathfrak{F}_{t-1}] \\ &= \hat{S}_{t-1}\exp(r_t - g(\sigma_t;\alpha,\tilde{\lambda}_+,\tilde{\lambda}_-))E_{\mathbb{Q}}[E_{\mathbf{Q}_t}[\exp(\sigma_t(\varepsilon_t+k))|\sigma_t]|\mathfrak{F}_{t-1}] \\ &= \hat{S}_{t-1}\exp(r_t - g(\sigma_t;\alpha,\tilde{\lambda}_+,\tilde{\lambda}_-))E_{\mathbb{Q}}[\exp(g(\sigma_t;\alpha,\tilde{\lambda}_+,\tilde{\lambda}_-))|\mathfrak{F}_{t-1}] \\ &= \hat{S}_{t-1}\exp(r_t)\end{aligned}$$

(b) Since $\mathrm{Var}_{\mathbb{Q}}(\varepsilon_t + k|\mathfrak{F}_{t-1}) \overset{\text{a.s.}}{=} 1 \overset{\text{a.s.}}{=} \mathrm{Var}_{\mathbb{P}}(\varepsilon_t|\mathfrak{F}_{t-1})$, we can prove the equality.

(c) Let $\xi_t = \varepsilon_t + k$. Then $\xi_t \sim stdMTS(\alpha, \tilde{\lambda}_+, \tilde{\lambda}_-)$ under $\mathbf{Q}_t$ for $1 \le t \le T$, and the following equality holds :

$$\begin{aligned}\log\left(\frac{S_t}{S_{t-1}}\right) &= r_t - d_t - g(\sigma_t;\alpha,\tilde{\lambda}_+,\tilde{\lambda}_-) + \sigma_t(\varepsilon_t + k) \\ &= r_t - d_t - g(\sigma_t;\alpha,\tilde{\lambda}_+,\tilde{\lambda}_-) + \sigma_t\xi_t.\end{aligned}$$

In the variance process, ε_{t-1} has to be replaced by $\xi_{t-1} - k$ in order to achieve the desired result. $\square$

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