Fractals or I.I.D.: Evidence of Long-Range Dependence and Heavy Tailedness from Modeling German Equity Market Returns

Wei Sun
Institute for Statistics and Mathematical Economics
University of Karlsruhe, Germany

Svetlozar Rachev*
Institute for Statistics and Mathematical Economics
University of Karlsruhe, Germany
&
Department of Statistics and Applied Probability
University of California, Santa Barbara, USA

Frank J. Fabozzi
Yale School of Management, USA

*Address correspondence to Prof. Dr. S. Rachev, Institute for Statistics and Mathematical Economics, Uni-Karlsruhe, Postfach 6980, 76128 Karlsruhe, Germany. Email: rachev@statistik.uni-karlsruhe.de. W. Sun's research was supported by grants from the Deutschen Forschungsgemeinschaft. S. Rachev's research was supported by grants from Division of Mathematical, Life and Physical Science, College of Letters and Science, University of California, Santa Barbara, and the Deutschen Forschungsgemeinschaft.

Fractals or I.I.D.: Evidence of Long-Range Dependence and Heavy Tailedness from Modeling German Equity Market Returns

Abstract

Several studies find that the return volatility of stocks tends to exhibit long-range dependence, heavy tails, and clustering. Because stochastic processes with self-similarity possess long-range dependence and heavy tails, Rachev and Mittnik (2000) suggest employing self-similar processes to capture these characteristics in return volatility modeling. In this paper, we find using high-frequency data that German stocks do exhibit these characteristics. Using one of the typical self-similar processes, fractional stable noise, we empirically compare this process with several alternative distributional assumptions in either fractal form or i.i.d. form (i.e., normal distribution, fractional Gaussian noise, generalized extreme value distribution, generalized Pareto distribution, and stable distribution) for modeling German equity market volatility. The empirical results suggest that fractional stable noise dominates these alternative distributional assumptions both in in-sample modeling and out-of-sample forecasting. Our findings suggest that models based on fractional stable noise perform better than models based on the Gaussian random walk, the fractional Gaussian noise, and the non-Gaussian stable random walk.

Key Words: fractional stable noise, heavy tails, long-range dependence, self-similarity, volatility modeling

JEL Classification: C41, G14

1. Introduction

Because return volatility estimates are key inputs in valuation modeling and trading strategies, considerable research in the financial econometrics literature has been devoted to return volatility modeling. The preponderance of empirical evidence from financial markets throughout the world fails to support the hypothesis that returns follow a Gaussian random walk.[1] In addition to the empirical evidence, there are theoretical arguments that have been put forth for rejecting both the Gaussian assumption and the random walk assumption. One of the most compelling arguments against the Gaussian random walk assumption is that markets exhibit a fractal structure. That is, markets exhibit a geometrical structure with self-similarity when scaled (see, Mandelbrot (1963, 1997)). As to this point, the normal distribution assumption and the random walk assumption cannot both be simultaneously valid for describing financial markets. It seems that the only way to explain fractal scaling is to abandon either the Gaussian hypothesis or the random walk hypothesis. By abandoning the Gaussian hypothesis, researchers end up with stable Paretian distributions.[2] The normal distribution is a special case with finite variance (details are discussed in Rachev and Mittnik (2000)). The implication of rejecting the random walk hypothesis is that researchers must accept that returns in financial markets are not independent but instead exhibit trends. Markets prone to trending have been characterized by long-range dependence and volatility clustering. Samorodnisky and Taqqu (1994) demonstrate that the properties of some self-similar processes can be used to model financial markets that are characterized as being non-Gaussian and non-random walk. Such financial markets have been stylized by long-range dependence, volatility clustering, and heavy tailedness.

Long-range dependence or long memory denotes the property of a time series to exhibit persistent behavior, i.e., a significant dependence between very distant observations and a pole in the neighborhood of the zero frequency of its spectrum.[3] Long-range dependence time series typically exhibit self-similarity. The stochastic processes with self-similarity are invariant in distribution with respect to changes of time and space scale. The scaling coefficient or self-similarity index is a non-negative number denoted by H, the Hurst parameter. If $\{X(t+h) - X(h), t \in T\} \stackrel{d}{=} \{X(t) - X(0), t \in T\}$ for all $h \in T$, the real-valued process $\{X(t), t \in T\}$ has stationary increments. A succinct expression of self-similarity is $\{X(at), t \in T\} \stackrel{d}{=} \{a^H X(t), t \in T\}$. The process $\{X(t), t \in T\}$ is called *H-sssi* if it is self-similar with index H and has stationary increments (see, Samorodnisky and Taqqu (1994) and Doukhan *et al.* (2003)).

In modeling return volatility, long-range dependence, volatility clustering, and heavy tailedness

[1]See, Fama (1963, 1965), Mandelbrot (1963, 1997), and Rachev and Mittnik (2000).

[2]To distinguish between a Gaussian and non-Gaussian stable distribution, the latter is usually referred to as stable Paretian distribution or Lévy stable distribution. Referring to it as a stable Paretian distribution highlights the fact that the tails of the non-Gaussian stable density have Pareto power-type decay; referring to it as a Lévy stable distribution recognizes the pioneering works by Paul Lévy in characterizing the non-Gaussian stable laws (see Rachev and Mittnik (2000)).

[3]Baillie (1996) provides a survey of the major econometric research on long-range dependence processes, fractional integration, and applications in economics and finance. Doukhan *et al.* (2003) and Robinson (2003) provide a comprehensive review of the studies on long-range dependence. Bhansali and Kokoszka (2006) review recent research on long-range dependence time series. Recent theoretical and empirical research on long-range dependence in economics and finance is provided by Rangarajan and Ding (2006) and Teyssiére and Kirman (2006). Sun *et al.* provide a review of long-range dependence research based on using intra-daily data.

should be treated simultaneously in order to obtain more accurate predictions. Rachev and Mittnik (2000) note that for modeling financial data, not only does model structure play an important role, but distributional assumptions influence the modeling accuracy. A distribution that is rich enough to encompass those stylized facts exhibited in return data is the stable distribution. Fama (1963), Mittnik and Rachev (1993a, 1993b), Rachev (2003), and Rachev *et al.* (2005) have demonstrated the advantages of stable distributions in financial modeling. Moreover, Taqqu and Samorodnitsky (1994), Rachev and Mittnik (2000), Rachev and Samorodnitsky (2001), Doukhan *et al.* (2003), and Racheva and Samorodnitsky (2003) have reported that long-range dependence, self-similar processes, and stable distribution are very closely related.

In this paper, we empirically investigate the return distribution of 27 German DAX stocks using intra-daily data under two separate assumptions regarding the return generation process (1) it does not follow a Gaussian distribution and (2) it does not follow a random walk. In our empirical study, we develop the ARMA-GARCH model based on these assumptions. Abandoning the Gaussian hypothesis, we analyze our data by employing an ARMA-GARCH model with several independent and identically distributed (i.i.d.) residuals following a normal distribution, stable distribution, generalized extreme value distribution, and generalized Pareto distribution. When we desert the random walk hypothesis but maintain the Gaussian hypothesis, we utilize an ARMA-GARCH model with fractional Gaussian noise. The ARMA-GARCH model with fractional stable noise is used when we drop the assumptions that the return distribution is Gaussian and follows a random walk. Using several goodness of fit criteria for evaluating both in-sample simulation and out-of-sample forecasting for a sample with 6,600 observations, we find that the ARMA-GARCH model with fractional stable noise outperforms the other models investigated. This finding suggests that such a model can capture the stylized facts better without considering either the Gaussian or random walk hypotheses. In other words, this result supports the hypotheses that return distributions in financial markets are better characterized as fractals rather than Gaussian random walks.

We organized the paper as follows. In Section 2, we introduce two self-similar processes: fractional Gaussian noise and fractional stable noise. The method for estimating the parameters in the underlying process is introduced in Section 3. In Section 4, methods for simulating fractional Gaussian noise and fractional stable noise are explained. The empirical results are reported in Section 5, where we compare the goodness of fit for both in-sample simulation and out-of-sample forecasting based on several criteria for the ARMA-GARCH model with fractional stable noise and with other distributions. We summarize our conclusions in Section 6.

2. Specification of the self-similar processes

Lamperti (1962) first introduced the semi-stable processes (which we today refer to as self-similar processes). Let T be either R, $R_+ = \{t : t \geq 0\}$ or $\{t : t > 0\}$. The real-valued process $\{X(t), t \in T\}$ has stationary increments if $X(t+a) - X(a)$ has the same finite-dimensional distrituions for all $a \geq 0$ and $t \geq 0$. Then the real-valued process $\{X(t), t \in T\}$ is self-similar with exponent of self-similarity H for any $a > 0$, and $d \geq 1$, $t_1, t_2, ..., t_d \in T$, satisfying:

$$\Big(X(at_1), X(at_2), ..., X(at_d)\Big) \stackrel{d}{=} \Big(a^H X(t_1), a^H X(t_2), ...a^H X(t_d)\Big). \quad (1)$$

2.1 Fractional Gaussian noise

For a given $H \in (0,1)$, there is basically a single Gaussian *H-sssi* [4] process, namely fractional Brownian motion (fBm), first introduced by Kolmogorov (1940). Mandelbrot and Wallis (1968) and Taqqu (2003) define fBm as a Gaussian *H-sssi* process $\{B_H(t)\}_{t \in R}$ with $0 < H < 1$. Mandelbrot and van Ness (1968) define the stochastic representation

$$B_H(t) := \frac{1}{\Gamma(H+\frac{1}{2})}\left(\int_{-\infty}^{0}[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}]dB(s) + \int_{0}^{t}(t-s)^{H-\frac{1}{2}}dB(s)\right), \tag{2}$$

where $\Gamma(\cdot)$ represents the Gamma function:

$$\Gamma(a) := \int_0^\infty x^{a-1}e^{-x}dx,$$

and $0 < H < 1$ is the Hurst parameter. The integrator B is ordinary Brownian motion. The principal difference between fractional Brownian motion and ordinary Brownian motion is that the increments in Brownian motion are independent while in fractional Brownian motion they are dependent. For fractional Brownian motion, Samorodnitsky and Taqqu (1994) define its increments $\{Y_j, j \in Z\}$ as fractional Gaussian noise (fGn), which is, for $j = 0, \pm 1, \pm 2, ...$, $Y_j = B_H(j-1) - B_H(j)$.

2.2 Fractional stable noise

While fractional Brownian motion can capture the effect of long-range dependence, it has less power to capture heavy tailedness. The existence of abrupt discontinuities in financial data, combined with the empirical observation of sample excess kurtosis and unstable variance, confirms the stable Paretian hypothesis identified by Mandelbrot (1963, 1983). It is natural to introduce the stable Paretian distribution in self-similar processes in order to capture both long-range dependence and heavy tailedness. Samorodinitsky and Taqqu (1994) introduce the α-stable *H-sssi* processes $\{X(t), t \in R\}$ with $0 < \alpha < 2$. If $0 < \alpha < 1$, the exponent of self-similarity are $H \in (0, 1/\alpha]$ and if $1 < \alpha < 2$, the exponent of self-similarity are $H \in (0,1)$. In addition, Cohen and Samorodnitsky (2006) show that with exponent $H' = 1 + H(1/\alpha - 1)$, process $\{X(t), t \in R\}$ is a well-defined symmetric α-stable ($S\alpha S$) process. It has stationary increments and is self-similar. They show that (1) for $0 < \alpha < 1$, a family of H'-*sssi* $S\alpha S$ processes with $H' \in (1, 1/\alpha)$ is obtained, (2) for $1 < \alpha < 2$, a family of H'-*sssi* $S\alpha S$ processes with $H' \in (1/\alpha, 1)$ is obtained, and (3) for $\alpha = 1$, a family of *1-sssi* $S\alpha S$ processes is obtained.

There are many different extensions of fractional Brownian motion to the stable distribution. The most commonly used is linear fractional stable motion (also called linear fractional Lévy motion), $\{L_{\alpha,H}(a,b;t), t \in (-\infty,\infty)\}$, which Samorodinitsky and Taqqu (1994) define as

$$L_{\alpha,H}(a,b;t) := \int_{-\infty}^{\infty} f_{\alpha,H}(a,b;t,x)M(dx), \tag{3}$$

where

$$f_{\alpha,H}(a,b;t,x) := a\left((t-x)_+^{H-\frac{1}{\alpha}} - (-x)_+^{H-\frac{1}{\alpha}}\right) + b\left((t-x)_-^{H-\frac{1}{\alpha}} - (-x)_-^{H-\frac{1}{\alpha}}\right), \tag{4}$$

[4] The abbreviation of "*sssi*" means self-similar stationary increments, if the exponent of self-similarity H is to be emphasized, then "*H-sssi*" is adopted.

and a, b are real constants. $|a|+|b|>1$, $0<\alpha<2$, $0<H<1$, $H\neq 1/\alpha$, and M is an α-stable random measure on R with Lebesgue control measure and skewness intensity $\beta(x)$, $x\in(-\infty,\infty)$ satisfying: $\beta(\cdot)=0$ if $\alpha=1$. They define linear fractional stable noises expressed by $Y(t)$, and $Y(t)=X_t-X_{t-1}$,

$$\begin{aligned} Y(t) &= L_{\alpha,H}(a,b;t)-L_{\alpha,H}(a,b;t-1) \\ &= \int_R\Big(a\Big[(t-x)_+^{H-\frac{1}{\alpha}}-(t-1-x)_+^{H-\frac{1}{\alpha}}\Big] \\ &+ b\Big[(t-x)_-^{H-\frac{1}{\alpha}}-(t-1-x)_-^{H-\frac{1}{\alpha}}\Big]\Big)M(dx), \end{aligned} \tag{5}$$

where $L_{\alpha,H}(a,b;t)$ is a linear fractional stable motion defined by equation (3), and M is a stable random measure with Lebesgue control measure given $0<\alpha<2$. Samorodinitsky and Taqqu (1994) show that the kernel $f_{\alpha,H}(a,b;t,x)$ is d-self-similar with $d=H-1/\alpha$ when $L_{\alpha,H}(a,b;t)$ is $1/\alpha$-self-similar. This implies $H=d+1/\alpha$ (see Taqqu and Teverovsky (1998) and Weron *et al.* (2005)).[5] In this paper, if there is no special indication, the fractional stable noise (fsn) is generated from a linear fractional stable motion.

3. Estimation in self-similar processes

3.1 Estimating the self-similarity parameter in fractional Gaussian noise

Beren (1994) discusses the Whittle estimator of the self-similarity parameter. For fractional Gaussian noise, Y_t, let $f(\lambda;H)$ denote the power spectrum of Y after being normalized to have variance 1 and let $I(\lambda)$ the periodogram of Y_t, that is

$$I(\lambda)=\frac{1}{2\pi N}\left|\sum_{t=1}^{N}Y_t\,e^{i\,t\,\lambda}\right|^2. \tag{6}$$

The Whittle estimator of H is obtained by finding $\hat{H}$ that minimizes

$$g(\hat{H})=\int_{-\pi}^{\pi}\frac{I(\lambda)}{f(\lambda;\hat{H})}d\lambda. \tag{7}$$

3.2 Estimating the self-similarity parameter in fractional stable noise

Stoev *et al* (2002) proposed the least-squares estimator of the Hurst index based on the finite impulse response transformation (FIRT) and wavelet transform coefficients of the fractional stable motion. A FIRT is a filter $v=(v_0,v1,...,v_p)$ of real numbers $v_t\in\Re, t=1,...,p$, and length $p+1$. It is defined for X_t by

$$T_{n,t}=\sum_{i=0}^{p}v_i\,X_{n(i+t)}, \tag{8}$$

where $n\geq 1$ and $t\in N$. The $T_{n,t}$ are the FIRT coefficients of X_t, that is, the FIRT coefficients of the fractional stable motion. The indices n and t can be interpreted as "scale" and "location". If

[5] Some properties of these processes have been discussed in Mandelbrot and Van Ness (1968), Maejima (1983), Maejima and Rachev (1987), Manfields *et al.* (2001), Rachev and Mittnik (2000), Rachev and Samorodnitsky (2001), Racheva and Samorodnitsky (2003), Samorodnitsky (1994, 1996, 1998), Samorodinitsky and Taqqu (1994), and Cohen and Samorodnitsky (2006).

$\sum_{i=0}^{p} i^r v_i = 0$, for $r = 0, ..., q-1$, but $\sum_{i=0}^{p} i^q v_i \neq 0$, then the filter v_i can be said to have $q \geq 1$ zero moments. If $\{T_{n,t}, n \geq 1, t \in N\}$ is the FIRT coefficients of fractional stable motion with the filter v_i that have at least one zero moment, Stoev *et al.* (2002) prove the following two properties of $T_{n,t}$: (1) $T_{n,t+h} \stackrel{d}{=} T_{n,t}$ and (2) $T_{n,t} \stackrel{d}{=} n^H T_{1,t}$, where $h, t \in N$, $n \geq 1$. We assume that $T_{n,t}$ is available for the fixed scales n_j, $j = 1, ..., m$ and locations $t = 0, ..., M_j - 1$ at the scale n_j, since only a finite number, say M_j, of the FIRT coefficients are available at the scale n_j.

Using these properties, we have

$$E \log |T_{n_j,0}| = H \log n_j + E \log |T_{1,0}|. \tag{9}$$

The left-hand side of equation (9) can be approximated by

$$Y_{\log}(M_j) = \frac{1}{M_j} \sum_{t=0}^{M_j-1} \log |T_{n_j,t}|. \tag{10}$$

Then we get

$$\begin{pmatrix} Y_{\log}(M_1) \\ \vdots \\ Y_{\log}(M_m) \end{pmatrix} = \begin{pmatrix} \log n_1 & 1 \\ \vdots & \vdots \\ \log n_m & 1 \end{pmatrix} \begin{pmatrix} H \\ E \log |T_{1,0}| \end{pmatrix} + \begin{pmatrix} \sqrt{M}\Big(Y_{\log}(M_1) - E \log |T_{n_1,0}|\Big) \\ \vdots \\ \sqrt{M}\Big(Y_{\log}(M_m) - E \log |T_{n_m,0}|\Big) \end{pmatrix}. \tag{11}$$

In short, we can express equation (11) as follows

$$Y = X\theta + \frac{1}{\sqrt{M}} \varepsilon. \tag{12}$$

Equation (10) shows that the self-similarity parameter H can be estimated by a standard linear regression of the vector Y against the matrix X. Stoev *et al.* (2002) explain this procedure.

3.3 Estimating the parameters of stable Paretian distribution

The stable distribution requires four parameters for complete description: an index of stability $\alpha \in (0, 2]$ (also called the tail index), a skewness parameter $\beta \in [-1, 1]$, a scale parameter $\gamma > 0$, and a location parameter $\zeta \in \Re$. Unfortunately no closed-form expression for the density function and distribution function of a stable distribution exist. Rachev and Mittnik (2000) give the definition of the stable distribution: A random variable X is said to have a stable distribution if there are parameters $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, $\gamma \geq 0$ and ζ real such that its characteristic function has the following form:

$$E \exp(i\theta X) = \begin{cases} \exp\{-\gamma^\alpha |\theta|^\alpha (1 - i\beta (\sin\theta) \tan \frac{\pi\alpha}{2}) + i\zeta\theta\} & if \quad \alpha \neq 1 \\ \exp\{-\gamma |\theta| (1 + i\beta \frac{2}{\pi} (\sin\theta) \ln |\theta|) + i\zeta\theta\} & if \quad \alpha = 1, \end{cases} \tag{13}$$

and

$$\operatorname{sign}\theta = \begin{cases} 1 & if \quad \theta > 0 \\ 0 & if \quad \theta = 0 \\ -1 & if \quad \theta < 0. \end{cases} \tag{14}$$

Stable density is not only support for all of $(-\infty, +\infty)$, but also for a half line. For $0 < \alpha < 1$ and $\beta = 1$ or $\beta = -1$, the stable density is only for a half line.

In order to estimate the parameters of the stable distribution, the maximum likelihood estimation (MLE) method given in Rachev and Mittnik (2000) can be employed. Given N observations, $X = (X_1, X_2, \cdots, X_N)'$, for the positive half line, the log-likelihood function is of the form

$$\ln(\alpha, \lambda; X) = N \ln \lambda + N \ln \alpha + (\alpha - 1) \sum_{i=1}^{N} \ln X_i - \lambda \sum_{i=1}^{N} X_i^{\alpha}, \tag{15}$$

which can be maximized using, for example, a Newton-Raphson algorithm. It follows from the first-order condition,

$$\lambda = N \left(\sum_{i=1}^{N} X_i^{\alpha} \right)^{-1} \tag{16}$$

that the optimization problem can be reduced to finding the value for α which maximizes the concentrated likelihood

$$\ln^*(\alpha; X) = \ln \alpha + \alpha \nu - \ln \left(\sum_{i=1}^{N} X_i^{\alpha} \right), \tag{17}$$

where $\nu = N^{-1} \Sigma_{i=1}^{N} \ln X_i$. The information matrix evaluated at the maximum likelihood estimates, denoted by $I(\hat{\alpha}, \hat{\lambda})$, is given by

$$I(\hat{\alpha}, \hat{\lambda}) = \begin{pmatrix} N\hat{\alpha}^{-2} & \sum_{i=1}^{N} X_i^{\hat{\alpha}} \ln X_i \\ \sum_{i=1}^{N} X_i^{\hat{\alpha}} \ln X_i & N\hat{\lambda}^{-2} \end{pmatrix}.$$

Rachev and Mittnik (2000) have shown that, under fairly mild conditions, the maximum likelihood estimates $\hat{\alpha}$ and $\hat{\lambda}$ are consistent and have asymptotically a multivariate normal distribution with mean $(\alpha, \lambda)'$. [6]

4. Simulation of self-similar processes

4.1 Simulation of fractional Gaussian noise

Paxson (1997) provides a method to generate fractional Gaussian noise by using the Discrete Fourier Transform of the spectral density. Bardet *et al.* (2003) describe a simulation procedure based on this method that overcomes some of the practical implementation issues. The procedure is:

1. Choose an even integer M. Define the vector of the Fourier frequencies $\Omega = (\theta_1, ..., \theta_{M/2})$, where $\theta_t = 2\pi t/M$ and compute the vector $F = f_H(\theta_1), ..., f_H(\theta_{M/2})$, where

   $$f_H(\theta) = \frac{1}{\pi} \sin(\pi H)\Gamma(2H + 1)(1 - \cos \theta) \sum_{t \in \aleph} |2\pi t + \theta|^{-2H-1},$$

   and $f_H(\theta)$ is the spectral density of fGn.

[6] Other methods for estimating the parameters of a stable distribution (i.e., the method of moments based on the characteristic function, the regression-type method, and the fast Fourier transform method) are discussed in Stoyanov and Racheva-Iotova (2004a, 2004b, 2004c).

2. Generate $M/2$ i.i.d. exponential (exp(1)) random variables $E_1, ..., E_{M/2}$ and $M/2$ i.i.d. uniform ($U[0,1]$) random variables $U_1, ..., U_{M/2}$.

3. Compute $Z_t = \exp(2i\pi U_t)\sqrt{F_t E_t}$, for $t = 1, ..., M/2$.

4. Form the M-vector: $\tilde{Z} = (0, Z_1, ...Z_{(M/2)-1}, Z_{M/2}, \overline{Z}_{(M/2)-1}, ..., \overline{Z}_1)$.

5. Compute the inverse FFT of the complex Z to obtain the simulated sample path.

4.2 Simulation of fractional stable noise

Replacing the integral in equation (5) with a Riemann sum, Stoev and Taqqu (2004) generate the approximation of fractional stable noise. They introduce parameters $n, N \in \aleph$ and express the fractional stable noise $Y(t)$ as

$$Y_{n,N}(t) := \sum_{j=1}^{nN} \Big((\frac{j}{n})_+^{H-1/\alpha} - (\frac{j}{n} - 1)_+^{H-1/\alpha} \Big) L_{\alpha,n}(nt - j), \tag{18}$$

where $L_{\alpha,n}(t) := M_\alpha((j+1)/n) - M_\alpha(j/n)$, $j \in \Re$. The parameter n is mesh size and the parameter M is the cut-off of the kernel function.

Stoev and Taqqu (2003) describe an efficient approximation involving the Fast Fourier Transformation (FFT) algorithm for $Y_{n,N}(t)$. Consider the moving average process $Z(m)$, $m \in \aleph$,

$$Z(m) := \sum_{j=1}^{nM} g_{H,n}(j) L_\alpha(m - j), \tag{19}$$

where

$$g_{H,n}(j) := \Big((\frac{j}{n})^{H-1/\alpha} - (\frac{j}{n} - 1)_+^{H-1/\alpha} \Big) n^{-1/\alpha}, \tag{20}$$

and $L_\alpha(j)$ is the series of i.i.d standard stable Paretian random variables. Since $L_{\alpha,n}(j) \stackrel{d}{=} n^{-1/\alpha} L_\alpha(j), j \in \Re$, equation (18) and (19) imply $Y_{n,N}(t) \stackrel{d}{=} Z(nt)$, for $t = 1, ..., T$. Let $\tilde{L}_\alpha(j)$ be the $n(N+T)$-periodic with $\tilde{L}_\alpha(j) := L_\alpha(j)$, for $j = 1, ..., n(N+T)$ and let $\tilde{g}_{H,n}(j) := g_{H,n}(j)$, for $j = 1, ..., nN$; $\tilde{g}_{H,n}(j) := 0$, for $j = nN+1, ..., n(N+T)$. Then

$$\{Z(m)\}_{m=1}^{nT} \stackrel{d}{=} \Big\{ \sum_{j=1}^{n(N+T)} \tilde{g}_{H,n}(j) \tilde{L}_\alpha(n - j) \Big\}_{m=1}^{nT}, \tag{21}$$

because for all $m = 1, ..., nT$, the summation in equation (19) involves only $L_\alpha(j)$ with indices j in the range $-nN \leq j \leq nT - 1$. Using a circular convolution of the two $n(N+T)$-periodic series $\tilde{g}_{H,n}$ and $\tilde{L}_\alpha$ computed by using their Discrete Fourier Transforms (DFT), the variables $Z(n)$, $m = 1, ..., nT$ (i.e., the fractional stable noise) can be generated.

5. Empirical analysis

Our empirical analysis involves comparing the performance of six ARMA-GARCH models with different kinds of residuals (i.e., residuals with forms of white noise, fractional Gaussian noise, fractional stable noise, stable distribution, generalized Pareto distribution, and generalized extreme value distribution). The analysis is performed for in-sample simulation and out-of-sample forecasting of the German DAX stock returns based on four goodness of fit criteria: the Kolmogorov-Simirnov distance, Anderson-Darling distance, Cramer Von Mises distance, and Kuiper distance.

5.1 Data and Methodology

Recent research suggests that high-frequency data better reflect the market microstructure and increase the level of statistical significance (for example, Bollerslev and Wright (2000) and Dacorogna *et at* (2000)). In this study, we investigate the high-frequency data at 1-minute frequency for 27 German DAX component stocks[7] from January 7, 2002 to December 19, 2003. We calculate the stock returns by

$$y_{i,t} = \log\Big(\frac{P_{i,t}}{P_{i,t-1}}\Big). \tag{22}$$

We will let N ($N = 220,050$) denote the length of the sample. The sub-sample series used for the in-sample analysis are randomly selected by a moving window with length T. Replacement is allowed in the sampling. Letting T_F denote the length of the forecasting series, we perform one-week ahead out-of-sample forecasting ($1 \leq T \leq T + T_F \leq N$). In the empirical analysis, sub-sample length (i.e., the window length) of $T = 10,000$ (approximately one month) was chosen for the in-sample simulation and $T_F = 2,250$ (approximately one week) for the out-of-sample forecasting. A total of 6,600 sub-samples (200 sub-samples for each stock index) were randomly created.

We define the ARMA-GARCH model for the conditional mean equation as:

$$y_t = \alpha_0 + \sum_{i=1}^{r} \alpha_i \, y_{t-i} + \varepsilon_t + \sum_{j=1}^{m} \beta_j \varepsilon_{t-j}. \tag{23}$$

Let $\varepsilon_t = \sigma_t \, u_t$, where the conditional variance of the innovations, σ_t^2, is by definition

$$Var_{t-1}(y_t) = E_{t-1}(\varepsilon_t^2) = \sigma_t^2. \tag{24}$$

The general GARCH(p,q) processes for the conditional variance of the innovation is then

$$\sigma_t^2 = \kappa + \sum_{i=1}^{p} \gamma_i \, \sigma_{t-i}^2 + \sum_{j=1}^{q} \theta_j \, \varepsilon_{t-j}^2. \tag{25}$$

In our analysis, ARMA(1,1)-GARCH(1,1) has been parameterized with different kinds of u_t, i.e. white noise, fractional Gaussian noise, fractional stable noise, stable distribution, generalized Pareto distribution, and generalized extreme value distribution.

The Kolmogorov-Simirnov (KS) distance, Anderson-Darling (AD) distance, Kuiper (K) distance, and Cramer Von Mises (CVM) distance are used as the criterion for the goodness of fit testing. Letting

[7] The data are from German Karlsruher Kapitalmarktdatabank (KKMDB). The DAX index consists of 30 stocks and the composition of the index changes every year. The database we developed included data from January 2002 to January 2004 for stocks that remained in the index over the entire period. Only 27 stocks satisfied that requirement and they are the ones used in this study.

$F_s(x)$ denote the empirical sample distribution and $\tilde{F}(x)$ the estimated distribution function, these measures are defined as follows:

$$KS = \sup_{x \in \Re} |F_s(x) - \tilde{F}(x)|, \tag{26}$$

$$AD = \sup_{x \in \Re} \frac{|F_s(x) - \tilde{F}(x)|}{\sqrt{\tilde{F}(x)(1 - \tilde{F}(x))}}, \tag{27}$$

$$K = \sup_{x \in \Re} \Big(F_s(x) - \tilde{F}(x) \Big) + \sup_{x \in \Re} \Big(\tilde{F}(x) - F_s(x) \Big), \tag{28}$$

and

$$CVM = \int_{-\infty}^{\infty} \Big(F_s(x) - \tilde{F}(x) \Big)^2 d\tilde{F}(x). \tag{29}$$

The major disadvantage of KS statistics is that it tends to be more sensitive near the center of the distribution than at the tails. AD statistics can overcome this. The reliability of testing the empirical distribution increases with the help of these two statistics, with KS distance focusing on the deviations around the median of the distribution and AD distance on the discrepancies in the tails (see Rachev and Mittnik (2000)).

5.2 Preliminary test

Table 1 shows the descriptive statistics of the returns of the 27 DAX stocks in our study. From the statistics reported in this table, it can be seen that excess kurtosis exists. Figure 1 shows the *Q-Q* plot of some stock returns. Notice that a concave departure from the straight line (exponential distribution) in the *Q-Q* plot is an indication of a heavy-tailed distribution (whereas a convex departure shows the light-tailed distribution).

The Hurst index $H \in (0, 1)$ is the index of self-similarity. For Gaussian processes with stationary increments, when

1. $H \in (0, 0.5)$, the increments of a process tend to have opposite signs and thus are more zigzagging due to the negative covariance.

2. $H \in (0.5, 1)$, the covariance between these two increments is positive and less zigzagging of the process.

3. $H = 0.5$, the covariance between this two increments is zero.

This can be restated as following: If the Hurst index is

1. less than 0.5, the process displays “anti-persistence” (i.e., positive excess return is more likely to be reversed and the performance in the next period is likely to be below the average, or in the contrary, negative excess return is more likely to be reversed and the performance in the next period is likely to be above the average).

2. greater than 0.5, the process displays “persistence” (i.e., positive excess return or negative excess return is more likely to be continued and the performance in the next period is likely to be the same as that in the current period).

3. equal to 0.5, the process displays no memory (i.e., the performance in the next period has equal probability to be below and above the performance in the current period).

For fractional stable processes, if the process has the index α ($0 < \alpha < 2$), when $H = 1/\alpha$ which corresponds to a process with independent increments. We say this process has no memory. When $H > 1/\alpha$, the process displays long-range dependence and when $H < 1/\alpha$, the process displays negative dependence. In addition, long-range dependence is only possible when $\alpha > 1$, since $H \in (0, 1)$ (see, Samorodnitsky and Taqqu (1994)).

In order to check long-range dependence in stock returns, we use the methods introduced in Sections 3.1 and 3.2 to estimate the Hurst index under the Gaussian and stable assumptions. We employed the MLE method explained in Section 3.3 to estimate the stable parameter. The results, reported in Table 1, indicate that the Hurst index does not have an estimated value of 0.5 if fractional Gaussian noise is assumed. This suggests the occurence of either long memory or short memory under the Gaussian assumption [8]. In Table 1, we can observe both fluctuation and long memory under the non-Gaussian stable assumption.

Engle (1982) proposes a Lagrange-multiplier test for ARCH phenomenon. A test statistic for ARCH of lag order q is given by

$$X_q \equiv nR_q^2$$

where R_q^2 is the non-centered goodness-of-fit coefficient of a qth order autoregression of the squared residuals taken from the original regression

$$\hat{u}_t^2 = \omega_0 + \omega_1 \hat{u}_{t-1}^2 + \omega_2 \hat{u}_{t-2}^2 + \cdots + \omega_q \hat{u}_{t-q}^2 + e_t, \tag{30}$$

where $\hat{u}$ is the residual in the original regression equation. Under the null hypothesis of the residuals of the original model being normally i.i.d., the ARCH statistic of lag order q follows a χ^2 distribution with q degrees of freedom:

$$\lim_{n\to\infty} X_q \sim \chi_q^2.$$

We use Engle's test to check whether ARCH effect occurs. In Table 2, we report the test statistics and the critical values to reject the null hypothesis that there is no ARCH effect at different lag orders. It is clear from the results reported in the table that an ARCH effect is exhibited in the return time series studied.

We use the Ljung-Box-Pierce Q-statistic based on the autocorrelation function to test serial correlation (i.e., the memory effect). The Q-statistic is

$$Q :\sim \chi_m^2 = N(N+2) \sum_{k=1}^{m} \frac{\rho_k^2}{N-k}, \tag{31}$$

where N denotes the sample size, m the number of autocorrelation lags included in the statistic, and ρ_k the sample autocorrelation at lag order k which is

$$\rho_k = \frac{\sum_{t=1}^{N-k} y_t y_{t+k}}{\sum_{t=1}^{N} y_t^2}. \tag{32}$$

[8] There are various extensions of the self-similarity property for generalized random processes, see Dobrushin (1979).

Ljung and Box (1978) show that the Q-statistic is following the asymptotic chi-square distribution.

The Ljung-Box-Pierce test results reported in Table 3 indicate that the hypothesis that there is no serial correlation can be rejected at different lags. We find that the memory effect occurs in the returns for each stock from a lag of 10 minutes to a lag of one month. In order to see when the memory effect vanishes, we compare the Q-statistic with its corresponding critical value. When the quotient of the Q-statistic and the corresponding critical value is less than 1, we cannot reject the null hypothesis that there is no serial correlation. The results reported in Table 4 show that all the stock returns exhibit serial correlation even after a half year. In 8 months, the memory effect vanishes for 8 stocks and in 10 months, the memory effect vanishes for 19 stocks. From Table 4, we see that the decay of autocorrelation is slow.

5.3 Results

The AD, KS, CVM, and Kuiper statistics were calculated for the six candidate distributional assumptions. The results of the descriptive statistics of the computed values for the four criteria for the in-sample study are reported in Table 5. As can be seen from this table, the ARMA-GARCH with fractional stable noise model exhibits a smaller mean value for all criteria than the other five models. That is, for the in-sample study, the ARMA-GARCH with fractional stable noise model has the best performance. We also perform one week ahead out-of-sample forecasting for stock returns. The results for the descriptive statistics of the computed four criteria, reported in Table 6, indicate that the ARMA-GARCH with fractional stable noise model exhibits a smaller mean value for all criteria than the other five models. This suggests that the ARMA-GARCH with fractional stable noise is better at forecasting than the other models studied.

6. Conclusions

There is considerable interest in the modeling of market volatility. Most models assume that residuals are independent and identically distributed and follow the Gaussian distribution. But the overwhelming empirical evidence does not support the hypothesis that financial asset returns can be characterized as Gaussian random walks. There are a number of arguments against both the Gaussian assumption and random walk assumption. One of the most compelling arguments against the Gaussian random walk is that there exist fractals in financal markets. In this paper, we empirically investigate 27 German DAX stocks sampled over two years at one minute frequency level under three separate assumptions regarding the return generation process (1) it does not follow a Gaussian distribution, (2) it does not follow a random walk, and (3) it does not follow a Gaussian random walk. When we model non-Gaussian random walk, we employ one of the self-similar processes (i.e., fractional stable noise) to capture the fractal structure in financial markets.

In our empirical analysis, we investigate the ARMA-GARCH model with six different forms of residuals in both fractal forms (i.e., fractional stable noise and fractional Gaussian noise) and i.i.d. forms (i.e., stable distribution, white noise, generalized Pareto distribution, and generalized extreme value distribution) for the modeling volatility of 27 German stocks. In-sample (one month) simulation and out-of-sample (one week) forecasting were empirically investigated. By using parameters estimated from the empirical series, we simulate an in-sample series for each stock return and one-step ahead forecasting series with these six modeling structures. Then we compared the goodness of fit for the generated series to the sampled series by adopting four criteria for the goodness of fit test (the Kolmogorov-Simirnov distance, Anderson-Darling distance, Cramer von Mises distance, and Kuiper distance). Based on a comparison of these criteria, the empirical evidence shows that the ARMA-GARCH model with fractional stable noise demonstrates a better performance in modeling volatility than other models. Our results also indicate that there exists a fractal structure in financial markets and the i.i.d. assumption in modeling is inappropriate for characterizing financial asset returns.

The empirical evidence suggests that stocks exhibit three characteristics: long-range dependence, heavy tailedness, and volatility clustering. Many studies have found that the stable distribution is a better description of financial returns because it can capture heavy tailedness and has a close relationship with long-range dependence. As a self-similar process, fractional stable noise can capture the reported stylized facts in financial return data. This finding should be taken into account in modeling volatility so that more accurate prediction might be realized by well-defined functional models.

References

[1] Baillie, R.T. (1996). Long memory processes and fractional integration in econometrics. *Journal of Econometrics*, vol. 73, No. 1, pp. 5-59.

[2] Bardet, J., G. Lang, G. Oppenheim, A. Philippe, and M. Taqqu (2003). Generators of long-range dependent processes: a survey, In *Theory and Applications of Long-range Dependence*, P. Doulkhan, G. Oppenheim, and M. Taqqu (eds.). Birkhäuser: Boston.

[3] Beran, J. (1994). *Statistics for Long-Memory Processes*, Chapman & Hall: New York.

[4] Bhansali, R. J., and P. S. Kokoszaka (2006). Prediction of long-memory time series: a tutorial review, in *Processes with Long-Range Correlations: Theory and Applications*, G. Rangarajan and M. Ding, eds., Springer: New York.

[5] Bollerslev, T., and J. H. Wright (2000). Semiparametric estimation of long-memory volatility dependencies: The role of high-frequency data. *Journal of Econometrics*, vol. 98, pp. 81-106.

[6] Cohen, S., and G. Samorodnitsky (2006). Random rewards, fractional Brownian local times and stable self-similar processes. *The Annals of Applied Probability*, vol. 16, No. 3, pp. 1432-1461.

[7] Dobrushin, R. L. (1979). Gaussian and their subordinated self-similar random generalized fields. *The Annals of Probability*, vol. 7, No. 1, pp. 1-28.

[8] Doukhan, P., G. Oppenheim, and M. S. Taqqu (2003). *Theory and Applications of Long-Range Dependence*, eds., Birkhäuser: Boston.

[9] Engle, R. F. (1982). Autoregressive conditional heteroskedasticity with estimates of the variance of U.K. inflation. *Econometrica*, vol. 50, No. 4, pp. 987-1008.

[10] Fama, E. (1963). Mandelbrot and the stable Paretian hypothesis. *Journal of Business*, vol. 36, pp. 420-429.

[11] —— (1965). The behavior of stock market prices. *Journal of Business*, vol. 38, pp. 34-105.

[12] —— and R. Roll (1971). Parameter estimates for symmetric stable distributions. *Journal of the American Statistical Association*, vol. 66, pp. 331-338.

[13] Hols, M., and C. de Vries (1991). The limiting distribution of extremal exchange rate returns. *Journal of Applied Econometrics*, vol. 6, No. 3, pp. 287-302.

[14] Hurst, H. (1951). Long-term storage capacity of reservoirs. *Transactions of the American Society of Civil Engineers*, vol. 116, pp. 770-808.

[15] —— (1955). Methods of using long-term storage in reservoirs. *Proceedings of the Institution of Civil Engineers*, Part I, pp. 519-577.

[16] Kolmogoroff, A. N. (1940). Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum. *Comptes Rendus (Doklady) Acad. Sci. URSS (N.S.)*, vol. 26, pp. 115-118.

[17] Lamperti, J. (1962). Semi-stable stochastic Processes. *Transactions of the American Mathematical Society*, vol. 104, pp. 62-78.

[18] Mandelbrot, B. B. (1963). New methods in statistical economics. *Journal of Political Economy*, vol. 71, pp. 421-440.

[19] —— (1997). *Fractals and Scaling in Finance*. Springer: New York.

[20] —— (1983). *The Fractal Geometry of Nature*. Freeman: San Francisco.

[21] —— and J. Van Ness (1968). Fractional Brownian motions, fractional noises and applications. *SIAM Review*, vol. 10, No. 4, pp. 422-437.

[22] Mandelbrot, B. B. and J. Wallis (1968). Noah, Joseph and operational hydrology. *Water Resources Research*, vol. 4, pp. 909-918.

[23] Maejima, M. (1983). On a class of self-similar processes. *Zeitschrift für Vahrscheinlichkeitstheorie und verwandte Gebiete*, vol. 62, pp. 235-245.

[24] —— and S. Rachev (1987). An ideal metric and the rate of convergence to a self-similar process. *Annals of Probability*, vol. 15, pp. 702-727.

[25] Mansfield, P., S. Rachev, and G. Samorodnitsky (2001). Long strange segments of a stochastic process and long-range dependence. *The Annals of Applied Probability*. vol. 11, pp. 878-921.

[26] Mittnik, S., and S. T. Rachev (1993a). Modeling asset returns with alternative stable models. *Econometric Reviews*, vol. 12, pp. 261-330.

[27] —— (1993b). Reply to comments on "Modeling asset returns with alternative stable models" and some extensions. *Econometric Reviews*, vol. 12, pp. 347-389.

[28] Mittnik, S., M. S. Paolella, S. T. Rachev (2000). Diagnosing and treating the fat tails in financial returns data. *Journal of Empirical Finance*, vol. 7, pp. 389-416.

[29] —— (2002). Stationary of stable power-GARCH processes. *Journal of Econometrics*, vol. 106, pp. 97-107.

[30] Paxson, V. (1997). Fast, approximate synthesis of fractional Gaussian noise for generating self-similar network traffic. *Computer Communication Review*, vol. 27, No. 5, pp. 5-18.

[31] Rachev, S. (2003). *Handbook of Heavy Tailed Distributions in Finance*, eds., Elsevier: Amsterdam.

[32] —— and S. Mittnik (2000). *Stable Paretian Models in Finance*. Wiley: New York.

[33] Rachev, S. and G. Samorodnitsky (2001). Long strange segments in a long range dependent moving average. *Stochastic Processes and their Applications*, vol. 93, pp. 119-148.

[34] Rachev, S., C. Menn and F. Fabozzi (2005). *Fat-Tailed and Skewed Asset Return Distributions*. Wiley: New Jersey.

[35] Racheva, B. and G. Samorodnitsky (2003). Long range dependence in heavy tailed stochastic processes. In *Handbook of Heavy Tailed Distributions in Finance*, S. Rachev (ed.). Elsevier: Amsterdam.

[36] Rangarajan, G., and M. Ding (2006). *Processes with Long-Range Correlations: Theory and Applications*, eds., Springer: New York.

[37] Robinson, P. M. (2003). *Time Series with Long Memory*, eds., Oxford University Press: New York.

[38] Samorodnitsky, G. (1994). Possible sample paths of self-similar alpha-stable processes. *Statistics and Probability Letters*, vol. 19, pp. 233-237.

[39] —— (1996). A class of shot noise models for financial applications. In *Proceeding of Athens International Conference on Applied Probability and Time Series. Volume 1: Applied Probability*, C. Heyde, Yu. V. Prohorov, R. Pyke, and S. Rachev, (eds.). Springer: Berlin.

[40] —— (1998). Lower tails of self-similar stable processes. *Bernoulli*, vol. 4, pp. 127-142.

[41] —— and M. S. Taqqu (1994). *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, Chapman & Hall/CRC: Boca Raton.

[42] Stoev, S., V. Pipiras and M. Taqqu (2002). Estimation of the self-similarity parameter in linear fractional stable motion. *Signal Processing*, vol. 82, pp. 1873-1901.

[43] Stoev, S., and M. Taqqu (2004). Simulation methods for linear fractional stable motion and FARIMA using the Fast Fourier Transform. *Fractals*, vol. 12, No. 1, pp. 95-121.

[44] Stoyanov, S., and B. Racheva-Iotova (2004a). Univariate stable laws in the field of finance—approximation of density and distribution functions. *Journal of Concrete and Applicable Mathematics*, vol. 2, No. 1, pp. 38-57.

[45] —— (2004b). Univariate stable laws in the field of finance—parameter estimation. *Journal of Concrete and Applicable Mathematics*, vol. 2, No. 4, pp. 24-49.

[46] —— (2004c). Numerical methods for stable modeling in financial risk management. In *Handbook of Computational and Numerical Methods*, S. Rachev (eds.). Birkhäuser: Boston.

[47] Taqqu, M. S. and V. Teverovsky (1998). On estimating the intensity of long-range dependence in finite and infinite variance time series. In *A Practical Guide to Heavy Tails*, R. Adler, R. Feldman, and M. Taqqu (eds.). Birkhäuser: Boston.

[48] Teyssiére, G., and A. P. Kirman (2006). *Long Memory in Economics*, eds. Springer: Berlin.

[49] Willinger, W., V. Paxson, and M. Taqqu (1998). Self-similarity and heavy tails: structural modeling of network traffic. In *A Practical Guide to Heavy Tails*, R. Adler, R. Feldman, and M. Taqqu (eds.). Birkhäuser: Boston.

[50] Weron, A., K. Burnecki, S. Mercik, and K. Weron (2005). Complete description of all self-similar models driven by Lévy stable noise. *Physical Review E*, vol. 71, 016113.

Table 1: Statistical characteristics of stocks in the study.

	mean	std	skewness	kurtosis	max	min	$\tilde{\alpha}$	$\tilde{H}_{FGN}$	$\tilde{H}_{fsn}$
Adidas	1.83E-07	0.0012	0.4041	94.9171	0.0670	-0.0327	0.7903	0.4791	1.3168
Allianz	-3.01E-06	0.0015	-0.5477	212.8600	0.0949	-0.0905	1.4222	0.5203	0.5616
BASF	2.65E-07	0.0012	1.2349	121.8300	0.0863	-0.0341	1.2247	0.4548	0.4589
BAYER	-1.29E-06	0.0015	0.8794	85.5980	0.0877	-0.0521	1.3998	0.5137	0.4947
BMW	-1.27E-07	0.0012	0.1213	58.6840	0.0459	-0.0463	1.2010	0.4946	0.4873
Commerzbank	-4.40E-07	0.0018	0.2352	48.3320	0.0539	-0.0577	1.1694	0.5337	0.5186
Daimler	-7.83E-07	0.0013	-0.2877	56.4390	0.0469	-0.0597	1.4160	0.4628	0.4621
Dt.Bank	-5.72E-07	0.0012	-0.6959	68.6110	0.0339	-0.0571	1.3938	0.5332	0.5082
Dt.Post	2.16E-07	0.0015	-0.3476	41.8600	0.0393	-0.0512	0.8724	0.4258	1.0786
Dt. Telecom	-8.79E-07	0.0015	-0.8822	100.8200	0.0557	-0.0913	1.8337	0.5293	0.5190
Eon	-4.01E-07	0.0011	-0.2422	26.6700	0.0234	-0.0334	1.2434	0.4214	0.4381
Fressenius	-7.16E-07	0.0017	5.0585	646.8212	0.1982	-0.0447	0.8294	0.5411	1.2408
Henkel	3.64E-08	0.0011	1.9018	261.9911	0.0977	-0.0389	0.7562	0.4115	1.2339
Hypovereinsbank	-1.92E-06	0.0019	-2.3888	236.3100	0.0694	-0.1475	1.2688	0.5469	0.5304
Infineon	-2.36E-06	0.0020	-0.7092	145.4146	0.0707	-0.1016	1.6018	0.5351	0.5451
Linde	-1.01E-07	0.0013	-0.2952	44.1141	0.0340	-0.0392	0.7968	0.4716	1.2696
Lufthansa	-4.10E-07	0.0016	0.5289	79.2296	0.0821	-0.0541	1.2694	0.4647	0.4647
Man	3.80E-08	0.0016	0.3009	65.3610	0.0754	-0.0461	0.7645	0.5372	1.3134
Metro	2.66E-06	0.0015	0.1496	63.0310	0.0477	-0.0455	1.1293	0.4806	0.5084
Muechenerrueck	-3.52E-06	0.0015	-0.4968	105.6100	0.07250	-0.0748	1.3203	0.5842	0.5559
RWE	-9.71E-07	0.0013	-0.1861	27.6080	0.0353	-0.0355	1.2270	0.3960	0.4400
SAP	3.54E-06	0.0011	-0.1718	128.7802	0.0577	-0.0427	1.3693	0.5481	0.5485
Schering	-2.45E-07	0.0012	-0.3756	64.1430	0.0299	-0.0553	1.1411	0.4299	0.4271
Siemens	2.74E-06	0.0011	0.1990	49.7927	0.0379	-0.0350	1.4426	0.5251	0.5275
ThyssenKrupp	2.18E-06	0.0015	-0.1627	34.7571	0.0373	-0.0419	1.1064	0.5019	0.5126
Tui	-1.59E-06	0.0019	0.0812	145.4800	0.1253	-0.1032	0.8074	0.4965	1.1906
Volkswagen	-5.18E-07	0.0013	-0.5010	60.3170	0.0502	-0.0584	1.1807	0.4908	0.5042

Table 2: ARCH-test for different lags at $\alpha = 0.05$.

$q =$	$1min$	$2min$	$5min$	$10min$	$15min$	$20min$	$25min$	$30min$	$60min$
Adidas	4403.1	6357.5	6530.4	6772.9	6821.2	6836.3	6894.1	6901.5	6986.5
Allianz	24.9	35.8	62.6	111.6	120.2	124.1	126.7	129.1	136.3
BASF	52.6	58.6	70.3	83.0	88.1	92.6	95.0	97.2	110.3
BAYER	2590.8	2595.1	2739.4	4511.9	4837.8	4840.4	4853.3	4881.6	5068.2
BMW	1211.3	1618.1	1780.2	1941.7	2030.9	2096.5	2129.6	2154.1	2262.1
Commerzbank	2595.7	2831.3	3693.9	4046.8	4109.8	4136.0	4181.2	4196.0	4264.1
Daimler	307.0	374.5	402.4	475.8	504.5	522.2	537.0	552.8	578.2
Dt.Bank	1172.6	1304.8	1504.4	1786.6	1911.8	1992.4	2073.5	2130.8	2325.2
Dt.Post	1552.0	3241.9	3390.5	3542.5	3601.2	3631.2	3661.7	3678.2	3702.1
Dt. Telecom	319.7	364.3	418.0	488.8	537.9	569.6	597.1	625.8	693.7
Eon	5050.6	5534.0	5896.9	6418.7	6668.7	6796.2	6940.1	7028	7251.6
Fressenius	9701.1	11503.0	12842.1	12981.0	13044.0	13083.0	13111.0	13124.0	13211.0
Henkel	131.2	139.6	148.1	185.4	194.2	196.0	197.4	198.9	211.5
Hypovereinsbank	75.9	90.2	100.9	121.6	127.9	154.2	154.6	163.1	179.7
Infineon	2477.1	2995.4	3056.6	3087.6	3101	3104.4	3113.1	3117.4	3128.6
Linde	3526.6	5312.5	5564.4	5797.4	5896.7	6048.9	6102.1	6146.3	6192.0
Lufthansa	1311.3	1605.0	3466.5	3519.5	3590.8	3626.7	3643.2	3649.3	3688.6
Man	3302.3	3890.1	4360.1	4530.6	4564.7	4572.9	4590.3	4600.9	4628.0
Metro	3654.2	4173.9	5454.1	5510.7	5551.1	5582.3	5597.6	5615.3	5652.3
Muechenerrueck	361.4	493.5	620.2	652.4	674.5	688.4	697.2	708.9	793.5
RWE	1747.6	2354.2	3049.4	3653.3	3875.2	3983.9	4042.6	4098.7	4284.3
SAP	351.0	377.9	410.2	448.5	466.4	483.8	496.7	502.7	515.4
Schering	2059.0	2335.1	2616.9	2763.7	2828.9	2865.3	2910.5	2937	2956.6
Siemens	732.8	896.3	1045.9	1178.5	1339.0	1398.2	1475.8	1513.6	1614.2
ThyssenKrupp	2237.2	3127.1	3399.1	3566.4	3671.6	3794	3824.2	3852.9	3892.2
Tui	154.4	251.8	299.5	331.4	346.1	358.9	366.9	374.6	395.7
Volkswagen	2682.0	3367.0	4020.8	4806.4	4906.4	4980.2	5020.3	5036.7	5114.0
Critical value	3.8415	5.9915	11.0705	18.307	24.996	31.4104	37.6525	43.773	67.5048

Table 3: Ljung-Box-Pierce Q-test statistic for different lags at $\alpha = 0.05$.

$k =$	$10min$	$30min$	$1hr$	$2hr$	$4hr$	$1day$	$1wk$	$2wk$	$1mon$
Adidas	2016.7	2083.8	2131.5	2194.1	2321.3	2579.8	4910.9	7783.8	15872.0
Allianz	309.4	371.7	421.8	514.8	694.8	987.5	3923.6	7490.2	18300.0
BASF	624.7	665.5	708.2	800.2	961.6	1319.7	4014.8	7174.5	16150.0
BAYER	572.4	650.2	745.0	865.1	1136.3	1506.8	4453.2	7758.6	17859.0
BMW	653.2	698.7	741.5	849.2	983.3	1279.4	3855.4	6854.6	15903.0
Commerzbank	1990.2	2013.7	2067.1	2166.1	2289.2	2613.1	5127.9	8440.4	17903.0
Daimler	1324.3	1376.4	1428.9	1479.6	1618.1	1895.0	4461.3	7311.3	15993.0
Dt.Bank	710.6	760.1	830.1	933.4	1080.0	1453.6	3993.9	7076.4	16047.0
Dt.Post	2030.7	2096.6	2132.0	2187.1	2310.3	2579.8	5042.4	7870.4	16261.0
Dt. Telecom	2031.5	2084.9	2170.6	2289.2	2436.1	2794.6	5652.3	9147.6	19491.0
Eon	822.6	866.2	924.4	1020.4	1147.9	1515.8	4351.1	7554.7	17160.0
Fressenius	2565.9	2640.8	2680.1	2765.3	2885.1	3154.0	5705.7	8722.2	17796.0
Henkel	1742.1	1823.5	1878.5	1959.9	2130.6	2477.5	5168.2	8111.4	16844.0
Hypovereinsbank	910.2	959.8	1046.2	1138.5	1277.7	1586.7	3931.8	6924.0	16185.0
Infineon	1454.0	1518.1	1555.4	1619.4	1789.7	2071.6	4489.3	7482.8	16435.0
Linde	1048.7	1097.6	1136.9	1203.0	1371.1	1662.1	3986.9	6690.9	14994.0
Lufthansa	2491.6	2544.4	2590.2	2659.1	2823.4	3087.4	5568.6	8543.5	17758.0
Man	1817.7	1866.1	1913.5	1991.6	2134.2	2419.6	4778.6	7874.1	16652
Metro	1339.1	1400.2	1426.3	1506.9	1692.5	1954.3	4319.2	7230.2	16405.5
Muechenerrueck	220.2	280.9	332.8	409.6	555.2	880.1	3708.7	7020.0	16338.1
RWE	942.2	989.4	1051.8	1146.8	1303.7	1626.3	4683.1	8025.7	17937.3
SAP	484.2	511.3	554.4	629.7	748.9	1060.7	3254.8	6141.3	14647.2
Schering	1421.1	1451.2	1486.2	1579.2	1746.6	2006.5	4599.5	7511.9	15945.0
Siemens	776.5	850.9	929.6	1003.3	1158.3	1481.5	4221.6	7596.7	16957.0
ThyssenKrupp	2623.9	2693.6	2733.4	2813.3	2959.5	3206.6	5407.0	8185.1	16618.0
Tui	1561.9	1599.7	1633.4	1734.4	1873.2	2166.8	4732.4	7751.7	17175.0
Volkswagen	369.7	394.97	459.9	526.6	674.9	987.9	3598.3	6671.3	15722.0
Critical value	18.3	43.8	79.1	146.6	277.1	532.1	2515.1	4962.3	12256.0

Table 4: Ljung-Box-Pierce Q-test statistic compared with corresponding critical values for different lags at $\alpha = 0.05$.

$k =$	$1day$	$1week$	$2weeks$	$1month$	$2months$	$4months$	$6months$	$8months$	$10months$
Adidas	4.8487	1.9526	1.5686	1.2951	1.1898	1.0947	1.0515	1.0239	1.0299
Allianz	1.8560	1.5600	1.5094	1.4932	1.3902	1.2348	1.0749	0.9917	0.9557
BASF	2.4802	1.5963	1.4458	1.3177	1.2552	1.1445	1.0452	0.9959	0.9838
BAYER	2.8319	1.7706	1.5635	1.4572	1.3652	1.2013	1.0657	1.0012	0.9773
BMW	2.4045	1.5329	1.3813	1.2976	1.2269	1.1416	1.0524	1.0108	0.9924
Commerzbank	4.9112	2.0389	1.7009	1.4607	1.3198	1.1917	1.0869	1.0351	1.0239
Daimler	3.5616	1.7738	1.4734	1.3049	1.2272	1.1339	1.0510	1.0076	0.9989
Dt.Bank	2.7320	1.5880	1.4260	1.3094	1.2349	1.1262	1.0277	0.9864	0.9787
Dt.Post	4.8486	2.0049	1.5860	1.3268	1.2082	1.0977	1.0330	1.0174	1.0270
Dt. Telecom	5.2522	2.2474	1.8434	1.5903	1.4769	1.2730	1.1243	1.0352	0.9920
Eon	2.8489	1.7300	1.5224	1.4001	1.3238	1.1677	1.0313	0.9758	0.9704
Fressenius	5.9277	2.2686	1.7577	1.4520	1.3370	1.2179	1.1164	1.0531	1.0139
Henkel	4.6563	2.0549	1.6346	1.3743	1.2953	1.1638	1.0696	1.0328	1.0240
Hypovereinsbank	2.9821	1.5633	1.3953	1.3206	1.2731	1.1032	1.0071	0.9755	0.9655
Infineon	3.8934	1.7850	1.5079	1.3410	1.2748	1.1783	1.0882	1.0239	0.9898
Linde	3.1239	1.5852	1.3483	1.2234	1.1806	1.1272	1.0777	1.0231	0.9914
Lufthansa	5.8025	2.2141	1.7217	1.4489	1.3122	1.2106	1.1114	1.0540	1.0301
Man	4.5475	1.9000	1.5868	1.3587	1.2489	1.1460	1.0739	1.0315	1.0143
Metro	3.6729	1.7173	1.4570	1.3385	1.2598	1.1772	1.0847	1.0289	0.9925
Muechenerrueck	1.6540	1.4746	1.4147	1.3331	1.2643	1.1484	1.0282	0.9816	0.9721
RWE	3.0566	1.8620	1.6173	1.4636	1.3523	1.2060	1.0769	1.0169	0.9907
SAP	1.9935	1.2941	1.2376	1.1951	1.1414	1.0983	1.0388	1.0089	0.9942
Schering	3.7711	1.8288	1.5138	1.3010	1.2165	1.1067	1.0378	1.0153	1.0121
Siemens	2.7844	1.6785	1.5309	1.3836	1.2824	1.1324	1.0167	0.9690	0.9722
ThyssenKrupp	6.0266	2.1498	1.6495	1.3559	1.2374	1.1373	1.0737	1.0488	1.0392
Tui	4.0724	1.8816	1.5621	1.4013	1.3218	1.2191	1.1020	1.0404	0.9927
Volkswagen	1.8567	1.4307	1.3444	1.2828	1.2223	1.1262	1.0303	0.9896	0.9771

Table 5: Summary of in-sample goodness of fit statistics for different models.

a. AD-statistic	AD_{mean}	AD_{std}	AD_{median}	AD_{max}	AD_{min}	AD_{range}
ARMA-GARCH-fGn	46.6768	54.3660	13.7335	55.8541	21.6282	34.2259
ARMA-GARCH-fsn	44.1625	53.2522	15.2382	64.6001	1.3917	63.2084
ARMA-GARCH-normal	46.7177	54.3751	13.7480	58.5747	21.0690	37.5057
ARMA-GARCH-stable	45.4108	53.5900	14.3638	95.2204	2.9886	92.2318
ARMA-GARCH-gev	46.6401	54.2656	13.7441	60.2271	21.0914	39.1357
ARMA-GARCH-gpd	51.2203	54.4755	20.2070	109.5363	3.5018	106.0244
b. KS-statistic	KS_{mean}	KS_{std}	KS_{median}	KS_{max}	KS_{min}	KS_{range}
ARMA-GARCH-fGn	0.4998	0.4992	0.0034	0.5285	0.4887	0.0398
ARMA-GARCH-fsn	0.4938	0.4965	0.0261	0.9725	0.2745	0.6980
ARMA-GARCH-normal	0.5003	0.4994	0.0043	0.5455	0.4893	0.0562
ARMA-GARCH-stable	0.5089	0.4974	0.0622	0.9725	0.3910	0.5815
ARMA-GARCH-gev	0.5000	0.4989	0.0057	0.5775	0.4825	0.0950
ARMA-GARCH-gpd	0.5698	0.5198	0.1335	1.0000	0.4165	0.5835
c. CVM-statistic	CVM_{mean}	CVM_{std}	CVM_{median}	CVM_{max}	CVM_{min}	CVM_{range}
ARMA-GARCH-fGn	449.5326	517.0503	203.7237	896.6917	82.9310	813.7607
ARMA-GARCH-fsn	445.2936	515.2956	202.2463	1473.6038	34.4169	1439.1868
ARMA-GARCH-normal	449.6701	517.3712	203.7739	889.0445	82.9532	806.0913
ARMA-GARCH-stable	454.7954	516.7259	230.4694	2985.6218	57.5066	2928.1152
ARMA-GARCH-gev	449.2438	517.0805	203.8510	886.1477	83.0288	803.1189
ARMA-GARCH-gpd	524.3399	521.2781	370.3320	1978.9581	52.5637	1926.3945
d. Kuiper-statistic	$Kuiper_{mean}$	$Kuiper_{std}$	$Kuiper_{median}$	$Kuiper_{max}$	$Kuiper_{min}$	$Kuiper_{range}$
ARMA-GARCH-fGn	0.9931	0.9937	0.0029	0.9985	0.9757	0.0227
ARMA-GARCH-fsn	0.9693	0.9862	0.0473	0.9985	0.5125	0.4860
ARMA-GARCH-normal	0.9931	0.9938	0.0029	0.9990	0.9750	0.0240
ARMA-GARCH-stable	0.9796	0.9877	0.0224	0.9990	0.6550	0.3440
ARMA-GARCH-gev	0.9913	0.9925	0.0048	0.9990	0.9570	0.0420
ARMA-GARCH-gpd	0.9696	0.9773	0.0287	1.0000	0.6505	0.3495

Table 6: Goodness of fit statistics for out-of-sample one week forecasting of different models.

a. AD-statistic	AD_{mean}	AD_{std}	AD_{median}	AD_{max}	AD_{min}	AD_{range}
ARMA-GARCH-fGn	30.1821	22.5228	13.6283	55.2241	21.5834	33.6407
ARMA-GARCH-fsn	27.6038	22.4110	12.2880	68.1149	1.2129	66.9021
ARMA-GARCH-normal	30.1927	22.5228	13.6421	59.2046	21.1361	38.0684
ARMA-GARCH-stable	28.8034	22.3886	13.0021	101.7023	2.6941	99.0082
ARMA-GARCH-gev	30.1205	22.5005	13.5541	59.8893	20.7111	39.1782
ARMA-GARCH-gpd	32.3273	23.9319	15.3931	108.9876	4.1084	104.8792
b. KS-statistic	KS_{mean}	KS_{std}	KS_{median}	KS_{max}	KS_{min}	KS_{range}
ARMA-GARCH-fGn	0.5018	0.5006	0.0049	0.5375	0.4905	0.0470
ARMA-GARCH-fsn	0.4965	0.4985	0.0278	0.9555	0.2820	0.6734
ARMA-GARCH-normal	0.5020	0.5010	0.0055	0.5615	0.4880	0.0735
ARMA-GARCH-stable	0.5105	0.4990	0.0617	0.9653	0.4049	0.5603
ARMA-GARCH-gev	0.5018	0.5005	0.0064	0.5705	0.4846	0.0858
ARMA-GARCH-gpd	0.5700	0.5210	0.1333	1.0000	0.4049	0.5951
c. CVM-statistic	CVM_{mean}	CVM_{std}	CVM_{median}	CVM_{max}	CVM_{min}	CVM_{range}
ARMA-GARCH-fGn	226.5630	92.4072	245.5856	950.0136	82.5743	867.4392
ARMA-GARCH-fsn	223.6907	91.6806	244.8920	1873.2304	34.0265	1839.2039
ARMA-GARCH-normal	226.6043	92.4969	245.5955	948.5973	82.7246	865.8726
ARMA-GARCH-stable	229.7204	92.0018	264.0086	2852.7912	77.7136	2775.0774
ARMA-GARCH-gev	225.5252	92.2819	243.0575	933.6036	82.4613	851.1423
ARMA-GARCH-gpd	248.8990	94.0770	282.0017	1929.6210	55.7156	1873.9054
d. Kuiper-statistic	$Kuiper_{mean}$	$Kuiper_{std}$	$Kuiper_{median}$	$Kuiper_{max}$	$Kuiper_{min}$	$Kuiper_{range}$
ARMA-GARCH-fGn	0.9935	0.9940	0.0032	1.0000	0.9715	0.0285
ARMA-GARCH-fsn	0.9698	0.9870	0.0481	0.9990	0.5362	0.4627
ARMA-GARCH-normal	0.9935	0.9940	0.0032	1.0000	0.9725	0.0275
ARMA-GARCH-stable	0.9801	0.9885	0.0231	0.9995	0.6876	0.3118
ARMA-GARCH-gev	0.9918	0.9930	0.0052	0.9995	0.9592	0.0402
ARMA-GARCH-gpd	0.9703	0.9780	0.0299	1.0000	0.6425	0.3575

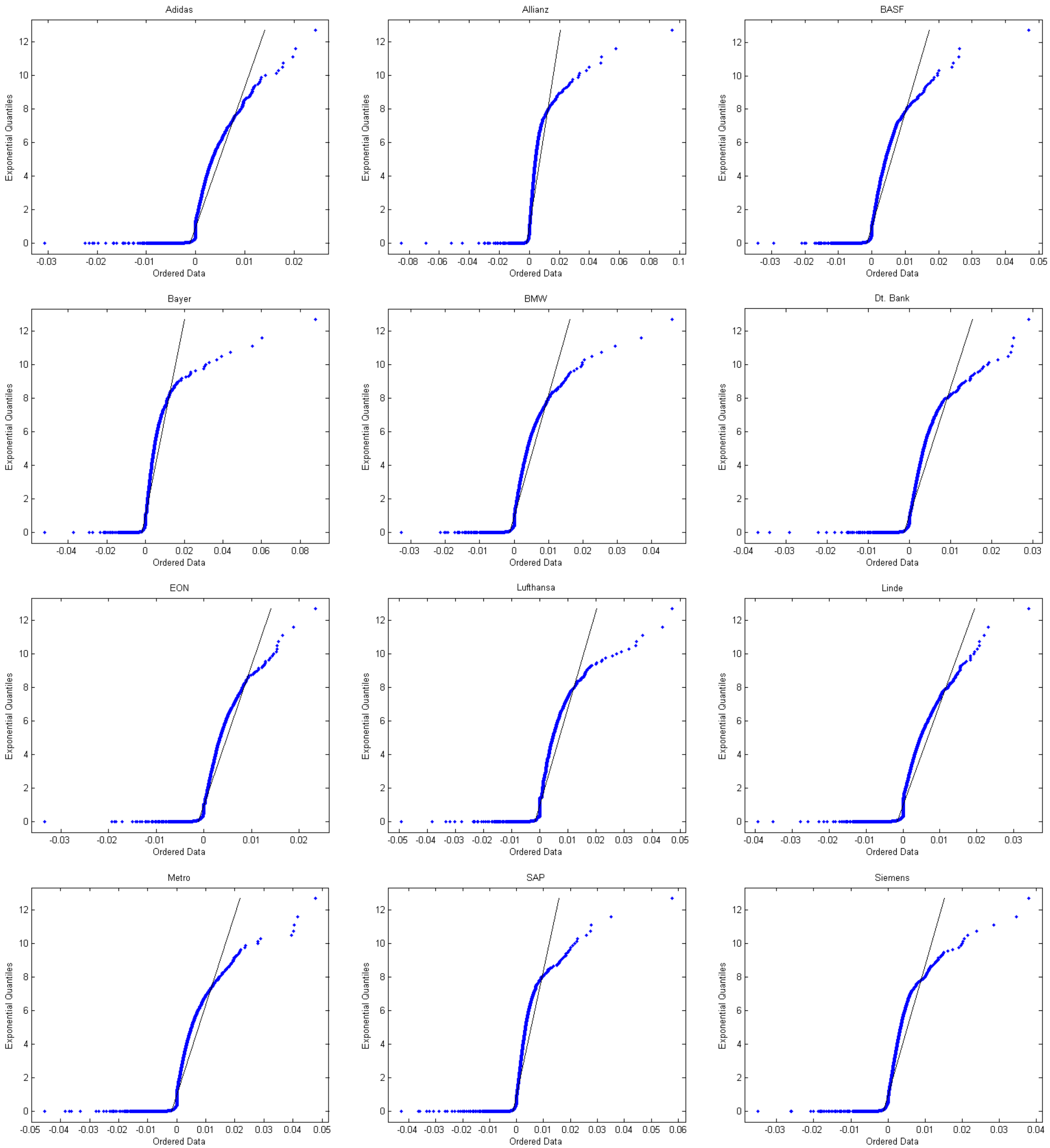

Figure 1: Q-Q plot of the returns for the stocks in study.