Stable distributions in the Black-Litterman approach to asset allocation

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Abstract. The integration of quantitative asset allocation models and the judgment of portfolio managers and analysts (i.e., qualitative view) dates back to papers by Black and Litterman (1990) [4], (1991) [5], (1992) [6]. In this paper we improve the classical Black-Litterman model by applying more realistic models for asset returns (the normal, the *t*-student, and the stable distributions) and by using alternative risk measures (dispersion-based risk measures, value at risk, conditional value at risk). Results are reported for monthly data and goodness of the models are tested through a rolling window of fixed size along a fixed horizon. Finally, we find that incorporation of the views of investors into the model provides information as to how the different distributional hypotheses can impact the optimal composition of the portfolio.

Key Words. Black-Litterman model, risk measures, return distributions.

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1 Introduction

The mean-variance model for portfolio management as formulated by Markowitz (1952) [20] is probably one of the most known and cited financial model. Despite its introduction in 1952, there are several reasons cited by academics and practitioners as why its use is not widespread. Some of the major reasons are the scarcity of diversification, see Green and Hollifield (1992) [14], or highly concentrated portfolios and the sensitivity of the solution to inputs (especially to estimation errors of the mean, see Kallberg and Ziemba (1981) [18], (1984) [19], Best and Grauer (1991) [3], Michaud (1989) [26]) and the approximation errors in the solution of the maximization problem.

The integration of quantitative asset allocation models and the judgment of portfolio managers and analysts (i.e., qualitative view) has been motivated by various discussions on increasing the usefulness of quantitative models for global portfolio management. The framework dates back to papers by Black and Litterman (1990) [4], (1991) [5], (1992) [6] that led to development of extensions of the framework proposed by members of both the academic and practitioner communities. Subsequent research has explained the advantages of this framework, what is now popularly referred to as Black-Litterman model (BL *model hereafter*), as well as the model's main characteristics [1].

The BL model contributes to the asset management literature in two distinct directions. The first direction is the idea that there should exist an equilibrium portfolio, with which one can associate an equilibrium distribution of the market. The equilibrium distribution summarizes neutral information and is not as sensitive to estimation risk as estimations that are purely based on time-series analysis. The second contribution is the process that twists the equilibrium distribution according to the practitioner's views. In particular, the BL model uses a Bayesian argument to perform this step, giving rise to a posterior distribution for the market. The computation of both the equilibrium portfolio and the posterior distribution rely on the assumption of a normal market. As far as the computation of the posterior distribution (under the non-normal assumption) is concerned, results have been obtained in Meucci (2006b) [25] and Meucci (2006a) [24].

In most of the papers mentioned above, there are explicit or implicit

[1] See the papers by Fusai and Meucci (2003) [11], Satchell and Scowcroft (2000) [34], He and Litterman (1999) [15] and the books by Litterman (2003)[17] and Meucci (2005) [23].

assumptions that returns on N asset classes are multivariate Gaussian distributed, an assumption consistent with other mainstream theories in finance such as the standard Black-Scholes model (1973) [7]. However, there are numerous empirical studies [2] that show that in many cases returns are quite far from being normally distributed, especially for high frequency data. Many recent papers (see Ortobelli, Huber, Rachev and Schwartz (2002) [28], Ortobelli, Rachev and Schwartz (2002) [29], Bertocchi, Giacometti, Ortobelli and Rachev (2005) [2]) show that stable Paretian distributions are suitable for the autoregressive portfolio return process in the framework of asset allocation problem over a fixed horizon.

As we pointed out above, there are two distinct directions in which the BL model contributes to the field of portfolio management. In this paper, we investigate further the first direction by exploring more generic distributional assumptions, namely in the computation of the equilibrium portfolio. We investigate whether the BL model can be enhanced by using the stable Paretian distributions as a statistical tool for asset returns. We use as a portfolio of assets a subset, duly constructed, of the S&P500 benchmark. We

[2] See Eberlein and Keller (1995) [8], Embrechts, Lindskog and McNeil (2003) [9], Rachev and Mittnik (2000) [32] and the references therein, Mittnick and Paolella (2000) [27], Panorska, Mittnik and Rachev (1995) [31], Tokat, Rachev and Schwartz (2003) [37], Tokat and Schwartz (2002) [36].

generalize the procedure of the BL model allowing the introduction of dispersion matrices obtained from multivariate Gaussian distribution, symmetric t-Student, and α-stable distributions for computing the equilibrium returns. Moreover, three different measures of risk (variance, value at risk and conditional value at risk) are considered. Results are reported for monthly data and goodness of the models are tested through a rolling window of fixed size along a fixed horizon. Results for weekly data are also available, however, the BL approach, which is a strategic asset allocation model, is usually adopted for at least monthly data. Finally, our analysis shows that the incorporation of the views of investors into the model provides information as to how the different distributional hypotheses can impact the optimal composition of the portfolio.

2 The α-stable distribution

The α-stable distributions describe a general class of distribution functions.

The α-stable distribution is identified by four parameters: the index of stability $\alpha \in (0, 2]$ which is the parameter of the kurtosis, the skewness parameter $\beta \in [-1, 1]$; $\mu \in \Re$ and $\gamma \in \Re^+$ which are, respectively, the location and the dispersion parameter. If X is a random variable whose distribution is α-stable, we use the following notation to underline the parameter dependence

$$X \stackrel{d}{=} S_{\alpha}(\gamma, \beta, \mu) \tag{1}$$

The stable distribution is normal, when $\alpha = 2$ and it is leptokurtotic when $\alpha < 2$. A positive skewness ($\beta > 0$) identifies distributions with right fat tails, while a negative skewness ($\beta < 0$) typically characterizes distributions with left fat tails. Therefore, the stable density functions synthesize the distributional forms empirically observed in the real data. The Maximum Likelihood Estimation (MLE) procedure used to approximate stable parameters is described by Rachev and Mittnik (2000) [32]). Unfortunately the density of stable distributions cannot be express in closed form. Thus, in order to value the density function, it is necessary to invert the characteristic function.

In the case where the vector $\mathbf{r} = [r_1, r_2, \ldots r_n]$ of returns is sub-Gaussian α-stable distributed with $1 < \alpha < 2$, then the characteristic function of $\mathbf{r}$ assumes the following form:

$$\Phi_{\mathbf{r}}(\mathbf{t}) = E(exp(i\mathbf{t}'\mathbf{r})) = exp(-(\mathbf{t}'\mathbf{V}\mathbf{t})^{\frac{\alpha}{2}} + i\mathbf{t}'E(\mathbf{r})) \tag{2}$$

For the dispersion matrix $\mathbf{V} = [v_{ij}^2]$ we use the following estimation $\tilde{\mathbf{V}} = [\tilde{v}_{ij}^2]$ (see Ortobelli, Rachev, Huber and Biglova (2004) [30] and Lamantia, Ortobelli and Rachev (2005) [16])

$$\tilde{v_{ij}^2} = (\tilde{v_{jj}})^{2-q} A(q) \frac{1}{T} \sum_{k=1}^{T} \tilde{r_{ik}} |\tilde{r_{ik}}|^{q-1} sgn(\tilde{r_{jk}}) \tag{3}$$

where $\tilde{r_{jk}} = r_{jk} - E(r_j)$ is the *k-th* centered observation of the *j-th* asset, $A(q) = \frac{\Gamma(1-\frac{q}{2})\sqrt{\pi}}{2^q \Gamma(1-\frac{q}{\alpha})\Gamma(\frac{q+1}{2})}$ and $1 < q < \alpha$

$$\tilde{v_{jj}} = (A(p) \frac{1}{n} \sum_{k=1}^{n} |\tilde{r_{jk}}|^p)^{\frac{2}{p}}, \quad 1 < p < 2 \tag{4}$$

where $\tilde{r_{jk}} = r_{jk} - E(r_j)$ is the *k-th* centered observation of the *j-th* asset.

3 The Black-Litterman model for asset allocation and our extension

As previously mentioned, the BL model overcomes the critical step of expected return estimation, using the equilibrium returns defined as the returns implicit in the benchmark. If the Capital Asset Pricing Model holds and if the market is in equilibrium, the weights based on market capitalizations are also the weights of the optimal portfolio. If the benchmark is a good proxy for the market portfolio, its composition is the solution of an optimization problem for a vector of unknown equilibrium returns. Moreover, the equilibrium returns provide a neutral reference point for asset allocation. Black and Litterman argue that the only sensible definition of neutral returns is the set of expected returns that would clear the market if all investors had identical views. In fact, an investor with neutral views should select a passive strategy, tracking the benchmark portfolio. The equilibrium returns $\mathbf{\Pi}$ of the stocks comprising the benchmark are obtained by solving the unconstrained maximization problem faced by an investor with quadratic utility function or assuming normally distributed returns

$$Max \quad \mathbf{\Pi}'\mathbf{x} - \frac{\lambda}{2}\mathbf{x}'\mathbf{\Sigma}\mathbf{x} \tag{5}$$

where $\mathbf{\Sigma}$ is the covariance matrix of our stocks' returns.

From Kuhn-Tucker conditions on (5), and solving the reverse optimization problem we get

$$\mathbf{\Pi} = \lambda\mathbf{\Sigma}\mathbf{x} \tag{6}$$

The expected return $E(\mathbf{r})$ is assumed to be normally distributed

$E(\mathbf{r}) \sim N(\mathbf{\Pi}, \tau\mathbf{\Sigma})$ with the covariance matrix proportional to the historical one, rescaled by a shrinkage factor; since uncertainty of the mean is lower than the uncertainty of the returns themselves, the value of τ should be close to zero.

An active asset manager can deviate from the benchmark tracking strategy, according to his or her economic reasoning in the tactical asset allocation. The major contribution of the BL model is to combine the equilibrium returns with uncertain views about expected returns. In particular, the optimal portfolio weights are moved in the direction of assets favored by the investor. The investor's views have the effect of modifying $E(\mathbf{r})$ according to the degree of uncertainty. The larger the uncertainty the lesser the deviation from the neutral views. To this aim the new vector of expected returns is computed minimizing the Mahalanobis distance between the expected returns $E(\mathbf{r})$ and the equilibrium returns which are additionally constrained by the investor's view on the expected return. This brings us to the following model:

$$min \quad (E(\mathbf{r}) - \mathbf{\Pi})' \tau\mathbf{\Sigma}(E(\mathbf{r}) - \mathbf{\Pi}) \tag{7}$$

s.t. constraint

$$\mathbf{P}E(\mathbf{r}) = \mathbf{q} + \mathbf{e} \tag{8}$$

where $\mathbf{P}$ is a matrix with each row corresponding to one view, $\mathbf{q}$ is the vector containing the specific investor views, and $\mathbf{e}$ a random vector of errors in the view. If all views are independent, the covariance matrix is diagonal. Its diagonal elements are collected in the vector $\mathbf{e}$. This formulation leads to the interpretation of using one view at time with a certain degree of uncertainty, i.e. scenario by scenario. The idea of seeing the market expected return distribution conditioned on investor's views as the solution to (7) and (8) is intuitive and quite general since it does not depend on the type of distribution, see also Chapter 10 in Zimmermann et al. (2002) [38]. Using Bayes rule, we know that it is possible to compute the distribution of the market conditioned on investor's views, see Meucci (2005) [23].

In our analysis we consider different problems of optimal allocation among n risky assets with returns $[r_1, r_2, \ldots, r_n]$ using different risk measures – variance, Value at Risk (VaR), and Conditional Value at Risk ($CVaR$). Assume that all portfolios $\mathbf{r}'\mathbf{x}$ are uniquely determined by the neutral mean $\mathbf{\Pi}'\mathbf{x}$ and by the risk measure $\rho()$ that is defined alternatively as the dispersion $\mathbf{x}^{'}\mathbf{V}\mathbf{x}$, the $VaR_\delta(\mathbf{r}^{'}\mathbf{x})$ and the $CVaR_\delta(\mathbf{r}^{'}\mathbf{x})$. This means that instead of (5) we have

$$Max \quad \mathbf{\Pi}^{'}\mathbf{x} - \frac{\lambda}{2}\rho(\mathbf{r}^{'}\mathbf{x}). \tag{9}$$

We recall here that $VaR_\delta(X)$ is implicitely defined by $P(X \leq -VaR_\delta(X)) = \delta$, i.e. the δ percentile of the probability density function of the random variable X such that the probability that the random variable assumes a value less than x is greater than δ, where δ represents, in this framework, the maximum probability of loss that the investor would accept. We also recall that $CVaR_\delta(X)$ is defined as $-E(X|X \leq -VaR_\delta(X))$, i.e it measures the expected value of the tail of the distribution for values less than VaR_δ. Note also that $CVaR_\delta(X)$ is a coherent risk measure in the sense of Artzner, Delbaen, Eber and Heath (1998) [1] while $VaR_\delta(X)$ is not.[3]

Notice that for elliptical distributions, following Embrechts,Lindskog and McNeil (2003) [9] and Stoyanov, Samorodnitsky, Rachev and Ortobelli (2004) [35], the $CVaR$ of portfolio returns is expressed as

[3]For detailed description of $CVaR$ see for example Rockafeller and Uryasev (2000) [33] and for α-stable see Stoyanov, Samorodnitsky, Rachev, Ortobelli (2004) [35]. For comparisons between $CVaR$ and VaR see Gaivoronski and Pflug (2005) [12].

$$CVaR_{\delta}(\mathbf{r}^{'}\mathbf{x}) = \sqrt{\mathbf{x}^{'}\mathbf{V}_t\mathbf{x}}CVaR_{\delta}(Y) - \mathbf{x}^{'}\mathbf{u} \tag{10}$$

where $CVaR_{\delta}(Y)$ for univariate t-distribution takes the following form

$$CVaR_{\delta}(Y) = \frac{\Gamma(\frac{\upsilon+1}{2})}{\Gamma(\frac{\upsilon}{2})}\frac{\sqrt{\upsilon}}{(\upsilon-1)\delta\sqrt{\pi}}(1+\frac{VaR_{\delta}(Y)^2}{\upsilon})^{\frac{1-\upsilon}{2}} \tag{11}$$

where Y is distributed according to a t-student with $\upsilon > 1$ degree of freedom.

Using again Stoyanov, Samorodnitsky, Rachev and Ortobelli (2004) [35], we can represent the $CVaR_{\delta}(\mathbf{X})$ for multivariate standardised α-stable distribution, $\mathbf{X} \in S_{\alpha}(\gamma,\beta,\mu)$ as

$$CVaR_{\delta}(\mathbf{X}) = \frac{\alpha}{1-\alpha}\frac{|VaR_{\delta}(\mathbf{X})|}{\pi\delta}\int_{-\theta_0}^{\frac{\pi}{2}} g(\theta)exp(-|VaR_{\delta}(\mathbf{X})|^{\frac{\alpha}{\alpha-1}}v(\theta))d\theta \tag{12}$$

where

$$g(\theta) = \frac{\sin(\alpha(\theta_0+\theta)-2\theta)}{\sin\alpha(\theta_0+\theta)} - \frac{\alpha\cos^2\theta}{\sin^2\alpha(\theta_0+\theta)}, \tag{13}$$

$$v(\theta) = (\cos\alpha\theta_0)^{\frac{1}{\alpha-1}}(\frac{\cos\theta}{\sin\alpha(\theta_0+\theta)})^{\frac{\alpha}{1-\alpha}}\frac{\cos(\alpha\theta_0+(\alpha-1)\theta)}{\cos\theta}, \tag{14}$$

and

$$\theta_0 = \frac{1}{\alpha}\arctan(\beta\tan\frac{\pi\alpha}{2}), \qquad \beta = -sgn(VaR_\delta(\mathbf{X}))\beta. \tag{15}$$

In the case where we have a non-standardized α-stable, we need to use the following transformation

$$CVaR_\delta(\gamma\mathbf{X}+\mu) = \gamma CVaR_\delta(\mathbf{X}) - \mu, \tag{16}$$

where

$$\gamma\mathbf{X}+\mu \in S_\alpha(\gamma,\beta,\mu). \tag{17}$$

Properties similar to (10) and (16) hold for VaR too (see Lamantia, Ortobelli and Rachev (2005)) [16].

Notice that the optimization problem for $CVaR$ risk measures using properties (10) and (16), can be written as

$$Max \quad \mathbf{\Pi}'\mathbf{x} - \frac{\lambda}{2}(CVaR_\delta\sqrt{\mathbf{x}'\mathbf{V}\mathbf{x}} - E(\mathbf{r})'\mathbf{x}) \tag{18}$$

Similar considerations apply to VaR_δ.

Applying first-order Kuhn-Tucker conditions to (18), the reverse optimization model, and using the three different measures of risks and the three different return distributions (Gaussian, t-student, stable) we obtain the following equilibrium returns for the three different dispersion measures $\mathbf{V}$ characterizing the three distributions:

Risk Measure: variance

$$\mathbf{\Pi} = \lambda \mathbf{V}\mathbf{x} \tag{19}$$

Risk Measure: CVaR

$$\mathbf{\Pi} = \frac{\lambda}{2}(CVaR_{\delta}\frac{\mathbf{V}\mathbf{x}}{\sqrt{\mathbf{x}'\mathbf{V}\mathbf{x}}} - E(\mathbf{r})) \tag{20}$$

Risk Measure: VaR

$$\mathbf{\Pi} = \frac{\lambda}{2}(VaR_{\delta}\frac{\mathbf{V}\mathbf{x}}{\sqrt{\mathbf{x}'\mathbf{V}\mathbf{x}}} - E(\mathbf{r})) \tag{21}$$

In formulas (19), (20), and (21), we will substitute the convenient estimate for the dispersion matrix, $CVaR$ and VaR depending on the corresponding distribution.

Notice that the coefficient λ can be interpreted as a coefficient of risk aversion: if λ is zero the investor is risk neutral, if $\lambda > 0$, the investor is risk averse because investments with large dispersion are penalized, if $\lambda < 0$, the investor is a risk seeker because investments with large dispersion are favored. Once we found the neutral returns implied in the benchmark, we wanted to test the goodness of these equilibrium returns over a 20 month horizon. We thought that a reasonable way was to compute the sum of squared errors between the neutral view return suggested by our model and the day after realization of return for 20 consecutive months, using a rolling window of 110 months for the parameters estimation. We compare the equilibrium returns obtained under different distributional hypotheses and different risk measures with a naive forecast: the unconditional mean. We recall that for a stationary return process the best forecast of future realizations is the unconditional mean.

But which is the optimal value of λ to be used? Black and Litterman suggest, under the normal distributional hypothesis, using the market risk premium which is 0.32 in our case. [4]

If we try to determine the value of λ which minimizes the distance between the optimal solution of the portfolio and the weight of the benchmark we get $\lambda = 36.29$, i.e. the risk aversion parameter becomes very large. This may be considered reasonable when we look at it from the equity premium puzzle side (see Fama and French (2002) [10], Mehra and Prescott (1985) [21], (2003) [22]). However since we consider three different risk measure we

[4] This is computed as the excess mean return divided by the variance.

must consider different values of λ. Indeed, in (19)-(21), the coefficient λ acts as a scaling factor, i.e. a larger λ increases the equilibrium returns. Large values of λ will eventually scale all the equilbrium return to very large values that would not be realistic.

We set the value of λ equal to the solution of an optimization problem. For each risk measure, we fix λ equal to the value that minimizes the sum of the squared error under all distributional hypothesis described before, computed for the first day of the out-of-sample analysis. We mantain the same values for all the out-of-sample analysis. Therefore, we choose $\lambda = 0.5$ for the case where risk was measured by dispersion and $\frac{\lambda}{2} = 0.15$ for the other cases. Finally, we tested how the forcing of a special investor view (both under certainty and under incertainty) may influence the benchmark composition.

4 Analysis of the data

In this section we analyze the time series data for the benchmark used in the computational part and we estimate the parameters of the different distributions (normal, symmetric t-student, and α-stable) and the related dispersion matrices. We selected as the benchmark the S&P500 and we obtained daily, weekly, and monthly data from July 1995 to July 2005 from DataStream. The analysis was done for all the frequencies mentioned above. However, only results for the monthly data are reported here.

We divided the data in two samples: the first 110 data for the parameter estimation and the remaining 20 data for out-of-sample analysis. The out-of-sample analysis is repeated for 20 consecutive months using a rolling window of length 110 to estimate the parameters for each month. In order to better analyze the results, we choose to reduce the dimension of the benchmark considering the most capitalized shares which account for about 50% of the index. We collected the data for the 50 most capitalized shares and rescaled the weights to sum up to 1. We use the new weights to construct a synthetic index that we will refer to as the S&P50 in the following. We also tested that S&P50 returns are almost perfectly correlated ($\rho = 0.98$) to the S&P500 along the considered horizon.

In Table 1 we report for each of the 50 shares included in the synthetic index, the ticker, the name of the company, the weight in the S&P500, the new weight in the S&P50, the mean, the volatility, the skewness, and the kurtosis of each share in the sample. Recall that we selected monthly data because in an asset allocation problem the reasonable time horizon should not be too short. Because of the frequency selected, we tested for the absence of autocorrelation in the returns and squared returns, but we found no evidence of it. Based on the Bera-Jarque test, we rejected for 19 of the 50 shares the null hypothesis of normal distribution at the 5% significance level, for 21 of the 50 at the 10% significance level, and for 23 of the 50 at the 15% significance level. From the results reported in Table 1 we can observe that about half of the 50 stocks could be well described by a normal distribution. In Table 2 we report the average of the estimated parameters α, β, γ for the α-stable distribution and the average of the degree of freedom for the t-student computed as the mean of 20 estimations over the rolling window. A similar analysis is done for a rolling window of increasing size, with no significant changes in the results. Therefore, we do not report those results. From Table 2 we note that only 9 stocks show a value of the α parameter equals to 2. The α-stable distribution looks more appropriate in describing the behavior of the remaining stocks returns.

In order to assess the hypothesis of non-normal behavior of the stocks and the statistical significance of the α-stable parameters, we estimated an autoregressive model on each estimated parameter of the α-stable distribution along the 20 consecutive months. Our reason for doing so is to have a statistical model that describes the evolution of the α-stable parameters over time. We then use the estimated statistical model to construct a confidence interval for the parameters. The following $\mathrm{AR}(1,1)$ model was estimated in order to construct a 90% confidence level for α and β

$$y_t = a_1 + a_2 y_{t-1} + \epsilon_t \sqrt{{a_3}^2} \tag{22}$$

where y_t is the α (or β) series, a_1 and a_2, a_3 the parameters to be estimated.

The estimated coefficients, together with their statistical significance, are reported in Table 3 for α and in Table 4 for β. We do not consider stocks with α=2 and β=1 or $\beta = \ -1$ for all the out-of-sample period. So we exclude 9 stocks from Table 3 and 17 stocks from Table 4. Each table contains the ticker, the estimated coefficients of the $\mathrm{AR}(1,1)$ process, the ratio between the value of the coefficients and the standard error, the value of the likelihood function, the 5-th, the 50-th, and the 95-th percentiles for the 40 stocks. We observe that the autoregressive coefficient is significant for 80% of the stocks for α and for 70% of the stocks for β (see columns T_{a_2}). We used the model given by (22) to create 5000 scenarios for each of the parameters α and β

and each stock in the benchmark, thus obtaining the related distributions: we report in Tables 3 and 4 the median, the 5-*th* and 95-*th* percentiles of those ones to costruct the 90% confidence level.

The analysis of the 90% confidence interval confirms that for 82% of the stocks considered the true value of α is less than 2, suggesting the presence of leptokurtic behavior. Only for 9 stocks is the normal distribution suitable. Moreover, the upper value of the confidence level of β is less then 0 for 19 stocks and β is equal to -1 for 6 stocks, suggesting a left fat tail distribution for 50% of the stocks. The lower value of the confidence level of β exceeds 0 for 7 stocks and β is equal to 1 for 2 stocks over the 50 stocks considered suggesting for a rigth fat tail for 18% of the stocks. Only for 7 stocks does the confidence interval include the null value suggesting that at most 16 stocks can show a symmetric behavior. That explains the poor behavior of a symmetric t-student distributional model which indeed seems to give the same result as the normal distribution. This is a further confirmation that the α-stable distribution is suitable to describe the returns of our data.

5 Computational results

In general under the normal and t-student distributions we get very similar equilibrium returns and portfolio composition under all the different risk measures while the α-stable hyphotesis implies different equilibrium returns and portfolio composition, see Giacometti, Bertocchi, Rachev and Fabozzi [13] for the detailed analysis.

We consider the equilibrium returns as a forecast of the future returns. Of course, we assume that when we compare the forecast with the future realizations, the data that we observe in the future are the products of a market in equilibrium. In Table 5 we report the sum of squared errors for 20 months between the neutral view and realization of the day after using a rolling window of 110 months. Note that we reestimate the parameters of the distribution as we move the rolling window. We observe that the hypothesis of a stable distribution and the use of dispersion as a risk measure gives the best combination [5]. For 13 months of the 20 months, it is the combination

[5]The use of a symmetric risk measure is not particularly surprising since we are dealing with a model of strategic allocation to find the optimal composition of our portfolio on a relative long time horizon. Generally this phase is followed by a tactical allocation strategy where it is more likely that a relative VaR can be considered by the market, i.e. the additional tail risk that we accept when we move from the benchmark replication strategy assuming specific risk.

that gives the best forecast (65% success rate). The second best combination is the α-stable distribution and the use of $CVaR$ as a risk measure. For this combination 3 of the 20 months gives the best forecast (15% success rate). The third best combination is the unconditional mean which resulted in 2 of the 20 months (10% success rate). Finally, following Satchell and Scowcroft (2000) [34], we compute the optimal composition for January 28, 2004 under a specific view for the different distributions and different risk measures with absolute certainty and with uncertainty.

We observe that the difference between the new returns and neutral view equilibrium are larger for the dispersion measure and normal returns, see Giacometti, Bertocchi, Rachev and Fabozzi [13] for the details. Indeed, this is the case with the highest variation in the portfolio composition. Once again the α-stable hypothesis with the same risk measure lead to a more stable portfolio. If we consider the same view with uncertainty, we have effects similar but mitigated by the uncertainty that we put in our view.

6 Conclusions

The purpose of our work is twofold. The first is to improve the classical BL model by applying more realistic models for asset returns. We compare the

BL model under the normal, t-student, and the stable distributions for asset returns. The second is to enhance the BL model by using alternative risk measures which are currently used in risk management and portfolio analysis. They include dispersion-based risk measures, value at risk, and conditional value at risk.

For the stocks in our sample, only a minority can be characterized as having a normal return distribution based on statistical tests we performed. As a result of incorporating heavy-tailed distribution models for asset returns and alternative risk measures, we obtained the following results: (1) the appropriateness of the α-stable distributional hypothesis is more evident when we compute the equilibrium returns and (2) the combination of α-stable distribution and the choice of dispersion risk measure provides the best forecast.

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ticker	company	% of $SP500$	% of $SP50$	*mean*	*volatility*	*skewness*	kurtosis	BJ (p-value)
XOM	Exxon Mobil Corp.	3.33	6.86	11.73%	0.90	−0.20	3.90	0.361
GE	General Electric	3.31	6.82	10.96%	0.60	0.12	2.93	0.870
MSFT	Microsoft Corp.	2.4	4.94	13.41%	1.24	−0.19	5.58	0.000
C	CityGroup Inc.	2.16	4.45	17.07%	1.08	−0.02	4.23	0.246
WMT	Wal-Mart Stores	1.81	3.73	12.15%	0.84	−0.39	3.28	0.128
PFE	Pfizer Inc.	1.81	3.73	11.46%	0.89	0	2.68	0.811
JNJ	Johnson & Johnson	1.73	3.56	11.97%	0.67	−0.08	3.16	0.930
BAC	Bank of America Corp.	1.62	3.34	10.79%	1.06	−0.41	4.46	0.057
INTC	Intel Corp.	1.45	2.99	10.85%	1.50	−0.88	5.74	0.000
AIG	American Intl Group	1.37	2.82	11.44%	0.87	0.01	3.40	0.600
MO	Altria Group Inc.	1.21	2.49	8.92%	1.11	−0.33	3.85	0.216
PG	Procter & Gamble	1.18	2.43	10.61%	0.85	−1.69	11.18	0.000
JPM	JPMorgan Chase & Co.	1.11	2.29	6.62%	1.25	−0.18	5.41	0.001
CSCO	Cisco Systems	1.09	2.24	16.87%	1.62	−0.63	4.60	0.000
IBM	Int. Business Machines	1.08	2.22	10.11%	1.17	0.37	4.86	0.003
CVX	Chevron Corp.	1.07	2.20	7.87%	0.63	0.38	3.31	0.140
WFC	Wells Fargo	0.93	1.92	13.53%	0.80	−0.05	3.82	0.737
KO	Coca Cola Co.	0.91	1.87	2.36%	0.84	−0.29	3.02	0.376
VZ	Verizon Communications	0.86	1.77	1.56%	0.83	0.00	3.32	0.990
DELL	Dell Inc.	0.85	1.75	33.87%	1.72	−0.66	4.89	0.000
PEP	PepsiCO Inc.	0.8	1.65	8.67%	0.73	−0.02	4.14	0.183
HD	Home Depot	0.76	1.57	13.12%	1.08	−0.67	4.38	0.001
COP	Conoco Philips	0.74	1.52	11.47%	0.76	0.18	3.32	0.839
SBC	SBC Communications Inc.	0.71	1.46	−0.09%	0.93	0.08	4.91	0.017
TWX	Time Warner Inc.	0.7	1.44	20.97%	1.96	0.43	3.72	0.340
UPS	United Parcel Service	0.69	1.42	6.86%	0.95	−0.60	4.39	0.008
ABT	Abbot Labs	0.68	1.40	8.89%	0.76	−0.32	3.70	0.162
AMGN	Amgen	0.68	1.40	17.35%	1.05	0.58	4.64	0.002
MRK	Merck & Co.	0.61	1.26	2.68%	1.00	−0.69	5.34	0.691
ORCL	Oracle Corp.	0.61	1.26	13.80%	1.68	−0.13	3.41	0.752
HPQ	Hewlett-Packard	0.61	1.26	4.03%	1.46	0.25	4.76	0.021
CMCSA	Compcast Corp.	0.6	1.24	10.99%	1.22	−0.06	5.52	0.000
UNH	United Health Group Inc.	0.6	1.24	20.27%	1.27	−2.31	14.11	0.000
AXP	Amercian Express	0.6	1.24	13.44%	0.94	−0.34	4.26	0.059
LLY	Lilly (Eli) & Co.	0.56	1.15	9.80%	0.97	0.03	3.48	0.654
MDT	Medtronic Inc.	0.56	1.15	15.04%	0.97	0.07	3.89	0.698
WYE	Wyeth	0.54	1.11	7.69%	1.13	−1.69	12.61	0.000
TYC	Tyco International	0.53	1.09	14.57%	1.43	−1.00	6.17	0.000
MWD	Morgan Stanley	0.52	1.07	13.5%	1.35	−0.33	3.85	0.174
FNM	Fannie Mae	0.51	1.05	8.61%	0.79	−0.36	2.98	0.280
MMM	3M Company	0.5	1.03	9.47%	0.71	0.20	3.17	0.559
QCOM	Qualcomm Inc.	0.49	1.01	23.8%	1.97	0.16	4.11	0.353
BA	Boeing Company	0.48	0.99	6.28%	1.04	−0.35	3.87	0.198
UTX	United Technologies	0.47	0.97	14.82%	0.89	−0.13	5.50	0.000
MER	Merryl Linch	0.47	0.97	13.20%	1.30	−0.36	4.80	0.007
VIA.B	Viacom Inc.	0.47	0.97	2.23%	1.11	0.08	3.10	0.870
DIS	Walt Disney Co.	0.46	0.95	2.81%	1.05	−0.01	3.40	0.940
G	Gillette Co.	0.45	0.93	3.94%	0.94	−0.50	4.05	0.0040
BMY	Bristol-Myers Squibb	0.44	0.91	7.94%	−0.41	−0.36	3.67	0.264
BLS	Bell South	0.44	0.91	4.32%	0.80	0.13	3.54	0.769

Table 1: Statistics on the single time series

Note: From the table we observe the skewness and kurtosis of the single stocks on the complete sample. We performe Bera-Jarque test and we cannot reject the null hypothesis of normal distribution for 19 over 50 at 5% significance level, for 21 over 50 at 10% significance level, and for 23 over 50 at 15% significance level.

numbering	ticker	company	% of $SP500$	% of $SP50$	α	β	γ	degree of freedom
1	XOM	Exxon Mobil Corp.	3.33	6.86	1.95	-0.91	0.64	18
2	GE	General Electric	3.31	6.82	2.00	0.96	0.42	100
3	MSFT	Microsoft Corp.	2.4	4.94	1.71	0.49	0.79	7
4	C	CityGroup Inc.	2.16	4.45	1.88	-0.18	0.74	10
5	WMT	Wal-Mart Stores	1.81	3.73	1.85	-1.00	0.58	22
6	PFE	Pfizer Inc.	1.81	3.73	2.00	-0.85	0.64	100
7	JNJ	Johnson & Johnson	1.73	3.56	2.00	-0.14	0.48	54
8	BAC	Bank of America Corp.	1.62	3.34	1.48	-0.44	0.58	4
9	INTC	Intel Corp.	1.45	2.99	1.81	-0.86	0.97	6
10	AIG	American Intl Group	1.37	2.82	2.00	-0.20	0.61	17
11	MO	Altria Group Inc.	1.21	2.49	1.90	-0.84	0.78	10
12	PG	Procter & Gamble	1.18	2.43	1.82	-1.00	0.51	4
13	JPM	JPMorgan Chase & Co.	1.11	2.29	1.69	-0.14	0.76	5
14	CSCO	Cisco Systems	1.09	2.24	1.77	-0.69	1.05	5
15	IBM	Int. Business Machines	1.08	2.22	1.77	0.39	0.73	6
16	CVX	Chevron Corp.	1.07	2.20	1.86	1.00	0.42	11
17	WFC	Wells Fargo	0.93	1.92	1.87	-0.01	0.56	8
18	KO	Coca Cola Co.	0.91	1.87	1.95	-1.00	0.59	92
19	VZ	Verizon Communications	0.86	1.77	2.00	-0.19	0.61	37
20	DELL	Dell Inc.	0.85	1.75	1.76	-0.69	1.08	5
21	PEP	PepsiCO Inc.	0.8	1.65	1.80	-0.02	0.47	6
22	HD	Home Depot	0.76	1.57	1.83	-1.00	0.72	9
23	COP	Conoco Philips	0.74	1.52	2.00	0.57	0.53	10
24	SBC	SBC Communications Inc.	0.71	1.46	1.81	-0.36	0.61	7
25	TWX	Time Warner Inc.	0.7	1.44	1.82	0.91	1.30	14
26	UPS	United Parcel Service	0.69	1.42	1.46	-0.45	0.51	4
27	ABT	Abbot Labs	0.68	1.40	1.94	-0.93	0.53	18
28	AMGN	Amgen	0.68	1.40	1.84	0.88	0.69	4
29	MRK	Merck & Co.	0.61	1.26	1.93	-1.00	0.67	16
30	ORCL	Oracle Corp.	0.61	1.26	2.00	-0.40	1.26	36
31	HPQ	Hewlett-Packard	0.61	1.26	1.74	0.05	0.94	5
32	CMCSA	Compcast Corp.	0.6	1.24	1.73	-0.16	0.75	4
33	UNH	United Health Group Inc.	0.6	1.24	1.54	0.15	0.54	3
34	AXP	Amercian Express	0.6	1.24	1.53	-0.84	0.54	8
35	LLY	Lilly (Eli) & Co.	0.56	1.15	1.94	-0.22	0.68	14
36	MDT	Medtronic Inc.	0.56	1.15	1.81	0.27	0.62	6
37	WYE	Wyeth	0.54	1.11	1.65	-0.56	0.62	4
38	TYC	Tyco International	0.53	1.09	1.71	-0.59	0.87	4
39	MWD	Morgan Stanley	0.52	1.07	1.82	-0.75	0.92	10
40	FNM	Fannie Mae	0.51	1.05	1.91	-1.00	0.54	100
41	MMM	3M Company	0.5	1.03	1.96	1.00	0.50	57
42	QCOM	Qualcomm Inc.	0.49	1.01	1.76	0.19	1.28	6
43	BA	Boeing Company	0.48	0.99	1.72	-0.42	0.67	6
44	UTX	United Technologies	0.47	0.97	1.67	-0.43	0.53	5
45	MER	Merryl Linch	0.47	0.97	1.81	-0.35	0.86	6
46	VIA.B	Viacom Inc.	0.47	0.97	2.00	-0.55	0.83	100
47	DIS	Walt Disney Co.	0.46	0.95	2.00	0.89	0.78	61
48	G	Gillette Co.	0.45	0.93	1.84	-0.87	0.65	7
49	BMY	Bristol-Myers Squibb	0.44	0.91	1.87	-0.97	0.60	7
50	BLS	Bell South	0.44	0.91	1.97	0.69	0.57	19

Table 2: Estimated parameters with α-stable and t-student distributions

Note: In this table we report the average estimate of the α-stable computed on a rolling window.

numbering	**ticker**	a_1	a_2	a_3	T_{a_1}	T_{a_2}	T_{a_3}	**LLF**	**perc.**5%	**perc.**50%	**perc.**95%
1	XOM	0.724	0.629	-	1.997	3.376	2.136	79.249	1.9357	1.9433	1.9507
2	GE	-	-	-	-	-	-	-	-	-	-
3	MSFT	0.349	0.793	0.005	0.831	3.497	3.406	25.658	1.4907	1.5982	1.7095
4	C	0.072	0.961	-	0.345	8.717	3.338	82.661	1.8551	1.8614	1.8680
5	WMT	0.361	0.805	-	1.130	4.677	1.543	70.703	1.8483	1.8598	1.8711
6	PFE	-	-	-	-	-	-	-	-	-	-
7	JNJ	-	-	-	-	-	-	-	-	-	-
8	BAC	0.232	0.842	-	0.962	5.265	2.959	47.765	1.4178	1.4538	1.4905
9	INTC	0.600	0.668	-	1.192	2.403	2.805	62.988	1.7805	1.7978	1.8144
10	AIG	-	-	-	-	-	-	-	-	-	-
11	MO	0.122	0.934	-	0.297	4.306	3.432	49.386	1.7882	1.8219	1.8567
12	PG	0.015	0.993	-	0.062	7.664	1.876	78.473	1.8378	1.8457	1.8534
13	JPM	0.187	0.889	-	0.612	4.949	3.775	59.791	1.6460	1.6667	1.6873
14	CSCO	0.827	0.532	-	2.981	3.400	2.789	48.111	1.7286	1.7640	1.7991
15	IBM	0.145	0.917	-	0.366	4.103	3.847	60.213	1.7101	1.7295	1.7491
16	CVX	0.991	0.468	-	3.667	3.203	3.582	64.867	1.8459	1.8610	1.8771
17	WFC	0.096	0.946	0.001	0.262	4.979	3.766	37.453	1.7114	1.7738	1.8348
18	KO	0.674	0.655	-	1.560	2.967	2.501	82.666	1.9445	1.9510	1.9575
19	VZ	-	-	-	-	-	-	-	-	-	-
20	DELL	0.025	0.980	0.003	0.031	2.158	2.347	29.273	1.4427	1.5345	1.6267
21	PEP	0.296	0.834	-	1.214	6.148	1.750	58.839	1.7261	1.7466	1.7680
22	HD	0.505	0.724	-	1.671	4.380	1.710	73.083	1.8217	1.8320	1.8421
23	COP	-	-	-	-	-	-	-	-	-	-
24	SBC	0.275	0.848	-	1.880	4.920	3.815	77.941	1.8063	1.8145	1.8228
25	TWX	0.191	0.894	0.001	0.434	3.700	3.648	46.213	1.7440	1.7825	1.8206
26	UPS	0.181	0.877	0.001	0.900	6.435	2.987	45.839	1.4429	1.4828	1.5227
27	ABT	1.853	0.045	-	0.620	0.029	1.511	57.538	1.9183	1.9411	1.9630
28	AMGN	0.695	0.622	0.002	1.501	2.358	2.351	33.662	1.7800	1.8543	1.9298
29	MRK	0.258	0.866	-	0.176	1.149	1.944	51.033	1.8728	1.9033	1.9341
30	ORCL	-	-	-	-	-	-	-	-	-	-
31	HPQ	0.133	0.922	-	0.519	6.250	3.758	57.519	1.6797	1.7025	1.7248
32	CMCSA	1.107	0.358	-	2.150	1.203	3.041	57.611	1.7001	1.7224	1.7452
33	UNH	0.412	0.733	0.001	0.791	2.153	2.913	43.604	1.4799	1.5252	1.5713
34	AXP	0.258	0.832	-	1.178	5.811	2.236	53.223	1.5128	1.5392	1.5675
35	LLY	0.168	0.912	0.001	0.411	4.361	3.423	44.314	1.8213	1.8654	1.9085
36	MDT	0.210	0.880	0.003	0.477	3.724	3.118	29.431	1.5760	1.6674	1.7598
37	WYE	0.484	0.706	-	0.806	1.943	3.730	51.040	1.6047	1.6354	1.6659
38	TYC	0.166	0.903	-	0.622	5.778	3.323	56.193	1.6705	1.6939	1.7178
39	MWD	0.675	0.630	0.001	1.041	1.808	1.784	40.896	1.7651	1.8172	1.8702
40	FNM	0.545	0.716	-	1.616	4.065	2.147	65.505	1.9010	1.9161	1.9321
41	MMM	0.456	0.767	-	1.552	5.105	2.480	73.090	1.9474	1.9578	1.9683
42	QCOM	0.268	0.847	0.001	0.619	3.463	3.689	44.868	1.6753	1.7170	1.7593
43	BA	0.647	0.624	0.001	2.402	3.948	2.584	45.355	1.6726	1.7132	1.7538
44	UTX	0.464	0.723	-	1.491	3.890	2.847	58.561	1.6539	1.6762	1.6973
45	MER	0.243	0.865	-	0.546	3.524	4.650	69.746	1.7862	1.7982	1.8103
46	VIA.B	-	-	-	-	-	-	-	-	-	-
47	DIS	-	-	-	-	-	-	-	-	-	-
48	G	1.100	0.401	-	1.505	1.011	2.053	55.033	1.8080	1.8329	1.8578
49	BMY	1.618	0.135	0.001	0.256	0.040	1.787	37.088	1.7882	1.8497	1.9112
50	BLS	1.299	0.339	-	2.196	1.143	1.333	54.315	1.9363	1.9633	1.9895

Table 3: Estimated parameters of the AR(1,1) process on α values

Note: The analysis of the 90% confidence interval confirm that for the 82% of the stocks considered (i.e. those with α different from 2) the true value of α is smaller than 2, suggesting the presence of leptokurtic behavior. Only for 9 stocks the normal distribution is suitable.

numbering	ticker	a_1	a_2	a_3	T_{a_1}	T_{a_2}	T_{a_3}	**LLF**	**perc.**5%	**perc.**50%	**perc.**95%
1	XOM	0.016	0.990	0.014	0.060	2.859	2.032	14.274	-0.5791	-0.3036	-0.1924
2	GE	-	-	-	-	-	-	-	-	-	-
3	MSFT	0.261	0.472	0.192	1.036	1.658	1.895	-11.859	-0.1468	0.5880	1.3106
4	C	-0.066	0.623	0.002	-1.289	2.198	3.043	35.618	-0.2072	-0.1405	-0.0746
5	WMT	-	-	-	-	-	-	-	-	-	-
6	PFE	-	-	-	-	-	-	-	-	-	-
7	JNJ	-	-	-	-	-	-	-	-	-	-
8	BAC	-0.014	0.959	-	-0.151	4.331	3.426	48.559	-0.4008	-0.3660	-0.3313
9	INTC	-0.015	0.975	0.001	-0.091	5.404	2.399	37.926	-0.7761	-0.7170	-0.6570
10	AIG	-	-	-	-	-	-	-	-	-	-
11	MO	-0.192	0.757	0.017	-0.720	2.353	1.905	12.488	-0.7972	-0.5816	-0.3709
12	PG	-	-	-	-	-	-	-	-	-	-
13	JPM	-0.037	0.737	0.001	-1.552	4.105	3.456	47.167	-0.1706	-0.1342	-0.0964
14	CSCO	-0.195	0.712	0.005	-1.527	4.227	2.171	24.582	-0.7508	-0.6355	-0.5192
15	IBM	0.004	0.966	0.003	0.041	3.872	4.751	30.636	0.1086	0.1927	0.2794
16	CVX	-	-	-	-	-	-	-	-	-	-
17	WFC	0.096	0.946	0.001	0.262	4.979	3.766	37.453	1.7114	1.7738	1.8348
18	KO	-	-	-	-	-	-	-	-	-	-
19	VZ	-	-	-	-	-	-	-	-	-	-
20	DELL	-0.011	0.936	0.051	-0.011	0.703	1.182	1.441	-0.3535	0.0161	0.3791
21	PEP	-0.008	0.599	0.001	-0.884	1.510	3.668	38.309	-0.0925	-0.0346	0.0262
22	HD	-	-	-	-	-	-	-	-	-	-
23	COP	-	-	-	-	-	-	-	-	-	-
24	SBC	-0.120	0.668	-	-1.930	3.780	3.230	47.633	-0.3885	-0.3516	-0.3138
25	TWX	0.199	0.785	0.005	0.782	2.638	2.198	23.887	0.8665	0.9836	1.1046
26	UPS	-0.190	0.577	-	-2.036	2.759	3.232	54.503	-0.4741	-0.4477	-0.4219
27	ABT	-0.977	-0.055	0.104	-0.056	-0.003	0.390	-5.739	-1.4412	-0.9204	-0.3926
28	AMGN	0.333	0.624	0.055	1.732	1.612	1.853	0.586	0.5667	0.9582	1.3432
29	MRK	-	-	-	-	-	-	-	-	-	-
30	ORCL	-	-	-	-	-	-	-	-	-	-
31	HPQ	0.012	0.728	0.001	0.848	3.593	1.909	43.827	-0.0299	0.0151	0.0604
32	CMCSA	0.005	0.980	0.002	0.146	5.190	5.146	34.298	-0.0683	0.0044	0.0790
33	UNH	0.026	0.860	0.003	0.405	2.431	3.941	29.404	0.1545	0.2446	0.3351
34	AXP	-0.179	0.788	-	-0.600	2.217	2.260	63.458	-0.8579	-0.8406	-0.8242
35	LLY	0.007	0.897	0.110	0.095	5.689	4.415	-6.319	-0.1457	0.4035	0.9520
36	MDT	0.196	0.259	0.006	3.042	1.386	2.744	23.535	0.1365	0.2571	0.3789
37	WYE	-0.176	0.687	0.001	-1.140	2.481	3.373	47.454	-0.6063	-0.5695	-0.5312
38	TYC	-0.036	0.930	0.002	-0.241	3.828	4.387	35.015	-0.5459	-0.4764	-0.4081
39	MWD	-0.255	0.653	0.013	-0.392	0.910	0.878	15.149	-0.8484	-0.6553	-0.4647
40	FNM	-	-	-	-	-	-	-	-	-	-
41	MMM	-	-	-	-	-	-	-	-	-	-
42	QCOM	0.034	0.808	0.001	0.815	3.773	3.859	37.019	0.0862	0.1486	0.2110
43	BA	-0.321	0.244	-	-2.092	0.681	2.442	57.426	-0.4422	-0.4200	-0.3975
44	UTX	-0.150	0.648	0.001	-1.570	2.889	2.430	45.302	-0.4441	-0.4042	-0.3628
45	MER	-0.049	0.852	0.001	-0.520	3.123	3.812	45.335	-0.3261	-0.2847	-0.2452
46	VIA.B	-	-	-	-	-	-	-	-	-	-
47	DIS	-	-	-	-	-	-	-	-	-	-
48	G	-0.249	0.708	0.004	-0.955	2.390	2.783	25.784	-0.8737	-0.7651	-0.6516
49	BMY	-1.972	-1.000	0.014	-0.001	-0.001	0.109	14.410	-1.7079	-1.5127	-1.3212
50	BLS	0.496	0.287	0.359	0.918	0.378	0.503	-18.137	-0.2001	0.7659	1.7537

Table 4: Estimated parameters of the AR(1,1) process on β values

Note: We can observe that, at 90% percent confidence level, the 50% of the stocks show a left fat tail, 18% a rigth fat tail. Since for 9 stocks, that account for 18% of the total, $\beta = 0$ by definition since $\alpha = 2$, and only for 7 stocks the confidence interval include the null value, we can suggest that at most 16 stocks can show a symmetric behavior.

	dispersion		VaR		CVaR		unc. mean
Date	**normal**	**stable**	**normal**	**stable**	**normal**	**stable**	
01/28/04	9.08	9.18	8.90	9.03	8.99	9.37	9.18
02/25/04	22.89	21.74	22.24	22.07	22.71	23.01	21.81
03/24/04	47.70	38.47	43.12	42.89	46.35	49.26	41.72
04/21/04	16.34	17.26	16.81	16.64	16.57	16.19	15.87
05/19/04	22.58	17.26	19.81	19.60	21.70	23.29	19.63
06/16/04	10.60	14.26	12.19	12.30	11.05	10.23	12.75
07/14/04	27.42	20.90	24.27	23.97	26.55	28.54	22.69
08/11/04	29.53	24.04	26.63	26.62	28.60	30.69	26.59
09/08/04	17.30	20.40	18.74	18.79	17.79	17.04	18.23
10/06/04	44.86	44.51	44.49	44.61	44.74	45.32	44.47
11/03/04	15.68	14.38	14.77	14.88	15.33	16.29	15.32
12/01/04	14.33	16.59	15.31	15.54	14.69	14.49	14.73
12/29/04	9.31	11.13	10.01	10.11	9.51	9.24	10.15
01/05/05	25.65	19.38	22.54	22.39	24.72	27.27	21.76
02/23/05	22.58	20.54	21.36	21.35	22.15	23.32	21.71
03/23/05	19.71	15.55	17.58	17.39	19.05	20.85	17.23
04/20/05	35.37	29.25	32.27	31.85	34.38	36.63	32.26
05/18/05	15.82	20.07	17.71	18.01	16.42	15.31	17.58
06/15/05	12.24	11.22	11.53	11.61	12.00	12.92	11.59
07/13/05	11.21	10.19	10.59	10.54	11.04	11.74	10.23

Table 5: Squared errors among the optimal composition and the unconditional mean over the rolling window horizon

Note:The hypothesis of stable distribution and the use of dispersion as risk measure gives the best combination. For 13 of the 20 considered months, it is the combination that gives the best forecast(65% success rate). The second best combination is the α-stable distribution and the use of $CVaR$ as risk measure. For this combination 3 of the 20 months gives the best forecast (15% success rate). The third best combination is the unconditional mean resulted in the best forecast in 2 of the 20 months(10% success rate).