Asymptotic distribution of the sample average value-at-risk

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Abstract

In this paper, we prove a result for the asymptotic distribution of the sample average value-at-risk (AVaR) under certain regularity assumptions. The asymptotic distribution can be used to derive asymptotic confidence intervals when $AVaR_\epsilon(X)$ is calculated by the Monte Carlo method which is adopted in many risk management systems. We study the effect of the tail behavior of the random variable X on the convergence rate and the improvement of a tail truncation method.

1 Introduction

The average value-at-risk (AVaR) risk measure has been proposed in the literature as a coherent alternative to the industry standard Value-at-Risk (VaR), see Artzner et al. (1998) and Pflug (2000). It has been demonstrated that it has better properties than VaR for the purposes of risk management and, being a downside risk-measure, it is superior to the classical standard deviation and can be adopted in a portfolio optimization framework, see Rachev et al. (2006), Stoyanov et al. (2007), Biglova et al. (2004), and Rachev et al. (2008).

The AVaR of a random variable X at tail probability ϵ is defined as

$$AVaR_\epsilon(X) = -\frac{1}{\epsilon}\int_0^\epsilon F^{-1}(p)dp.$$

where $F^{-1}(x)$ is the inverse of the cumulative distribution function (c.d.f.) of the random variable X. The random variable may describe the return of stock, for example. A practical problem of computing portfolio AVaR is that usually we do not know explicitly the c.d.f. of portfolio returns. In order to solve this practical problem, the Monte Carlo method is employed. The returns of the portfolio constituents are simulated and then the returns of the portfolio are calculated. In effect, we have a sample from the portfolio return distribution which we can use to estimate AVaR. The larger the sample, the closer the estimate to the true value. However, with any finite sample, the sample AVaR will fluctuate about the true value and, having only a sample estimate, we have to know the probability distribution of the sample AVaR in order to build a confidence interval for the true value. The sample AVaR equals,

$$\widehat{AVaR}_\epsilon(X) = -\frac{1}{\epsilon}\int_0^\epsilon F_n^{-1}(p)dp.$$

where $F_n^{-1}(p)$ denotes the inverse of the sample c.d.f. $F_n(x) = \frac{1}{n}\sum_{i=1}^n I\{X_i \leq x\}$ in which $I\{A\}$ denotes the indicator function of the event A.

The problem of computing the distribution of the sample AVaR is a complicated one even if we know the distribution of X. From a practical viewpoint, X describes portfolio returns which can be a complicated function of the joint distribution of the risk drivers. Therefore, we can only rely on large sample theory to gain insight into the fluctuations of sample AVaR. That is, for a large n, we can use a limiting distribution to calculate a confidence interval.

In this paper, first we prove a limit theorem for the sample AVaR in Section 2 . The limit theorem does not give answers to the question of how many simulations are necessary in order for the limiting distribution to be acceptable as a model for practical purposes. This number depends also on the distribution of X. A major factor is the tail behavior of X and, more precisely, how heavy the left tail of the distribution is. We study this problem in Section 3.1 assuming that X has Student's t distribution. Finally, we illustrate the impact of a tail truncation method in a finite and infinite variance case.

2 A limit theorem

In this section, we prove the following limit theorem.

Theorem 1. *Suppose that X is random variable with finite second moment $EX^2 < \infty$. Furthermore, suppose that the c.d.f. of X is differentiable at $x = q_\epsilon$, where q_ϵ is the ϵ-quantile of X. Then, as $n \to \infty$,*

$$\frac{\sqrt{n}}{\sigma_\epsilon}\left(\widehat{AVaR}_\epsilon(X) - AVaR_\epsilon(X)\right) \xrightarrow{w} N(0,1) \tag{1}$$

where $\xrightarrow{w}$ *denotes weak limit and*

$$\sigma_\epsilon^2 = \frac{1}{\epsilon^2} D(\max(q_\epsilon - X, 0)). \tag{2}$$

Proof. We apply the following more general result,

$$\phi(F_n) - \phi(F) \xrightarrow{w} N(0, \lambda^2)$$

where ϕ is a differentiable functional, F_n is the empirical c.d.f., F is the c.d.f. of X, and

$$\lambda^2 = D(\phi'(\delta_{X_i} - F)) = \int_R (IF_\phi(x))^2 dF(x) < \infty$$

in which IF_ϕ stands for the influence function of the functional ϕ[1], δ_{X_i} is the cdf of the observation X_i.[2] By the definition of the influence function,

$$\phi'(\delta_{X_i} - F) = \frac{d}{dt}(\phi((1-t)F + t\delta_{X_i}))|_{t=0} = \frac{d}{dt}(\phi(F_t))|_{t=0}.$$

The proof of the main result reduces to calculating the influence function of $\phi(F)$ and then calculating the variance λ^2. We need the assumed properties of the c.d.f. for the calculation of the influence function. In our case, from the definition of AVaR,

$$\begin{aligned} \phi(F) &= -\frac{1}{\epsilon}\int_0^\epsilon F^{-1}(p)dp \\ &= -F^{-1}(\epsilon) + \frac{1}{\epsilon}\int_{-\infty}^{F^{-1}(\epsilon)} F(p)dp. \end{aligned} \tag{3}$$

The influence function can be directly calculated,

$$\begin{aligned} IF_\phi(x) &= \frac{d}{dt}(\phi(F_t))|_{t=0} \\ &= -\frac{d}{dt}(F_t^{-1}(\epsilon))|_{t=0} + \frac{1}{\epsilon}\frac{d}{dt}\left(\int_{-\infty}^{F_t^{-1}(\epsilon)} F_t(p)dp\right)\Bigg|_{t=0} \end{aligned}$$

[1]Alternatively, the influence function can be regarded as the Gateaux derivative of ϕ in the direction of δ_{X_i}

[2]It can also be verified that $E(IF_\phi(X)) = 0$. In fact, this condition should hold from the general considerations behind the more abstract result.

The second term is differentiated separately below

$$\frac{d}{dt}\left(\int_{-\infty}^{F_t^{-1}(\epsilon)} F_t(p)dp\right)\Bigg|_{t=0} = \epsilon\frac{d}{dt}(F_t^{-1}(\epsilon))|_{t=0} + \max(q_\epsilon - x, 0) - \int_{-\infty}^{q_\epsilon} F(y)dy$$

where where q_ϵ stands for the ϵ-quantile of X and we take advantage of the chain rule

$$\frac{d}{dt}\left(\int_a^{f(t)} G(t,y)dy\right) = G(t, f(t))f'(t) + \int_a^{f(t)} G_t(t,y)dy$$

in which $f(x)$ is a monotonically increasing function. In computing the derivative we used that $F(x)$ is differentiable at $x = q_\epsilon$. Finally, for the influence function we obtain

$$IF_\phi(x) = \frac{1}{\epsilon}\max(q_\epsilon - x, 0) - \frac{1}{\epsilon}\int_{-\infty}^{q_\epsilon} F(y)dy$$

Now we can calculate the variance,

$$\lambda^2 = D(IF_\phi(X)) = \frac{1}{\epsilon^2}D(\max(q_\epsilon - X, 0)).$$

It is also straightforward to check that $E(IF_\phi(X)) = 0$,

$$E(IF_\phi(X)) = \frac{1}{\epsilon}E\max(q_\epsilon - X, 0) - \frac{1}{\epsilon}\int_0^\epsilon pdF^{-1}(p) = 0$$

follows after integration by parts. □

The variance of the asymptotic normal distribution is not possible to calculate if we do not know the cdf $F(x)$ of X. Therefore, if we have only a sample of i.i.d. observations, the variance σ^2 has to be estimated. To this end, expressing the variance in terms of conditional moments may be more useful. The variance of the asymptotic normal distribution given in (2) equals

$$\sigma_\epsilon^2 = \frac{q_\epsilon^2}{\epsilon} - \frac{2q_\epsilon}{\epsilon}E(X|X \leq q_\epsilon) + \frac{1}{\epsilon}E(X^2|X \leq q_\epsilon) - (q_\epsilon - E(X|X \leq q_\epsilon))^2 \quad (4)$$

An estimate of σ_ϵ^2 can be obtained by estimating the conditional moments and the corresponding quantile from the sample.

Furthermore, we would like to remark on a consistency with the classical theory behind constructing confidence intervals for the mean of a random variable. Suppose that the tail probability approaches one. In this case, the AVaR turns into the mean of X,

$$\lim_{\epsilon \to 1} AVaR_\epsilon(X) = EX,$$

the sample AVaR turns into the sample average,

$$\lim_{\epsilon \to 1} \widehat{AVaR}_\epsilon(X) = \frac{1}{n}\sum_{i=1}^{n} X_i,$$

where $X_1, \dots, X_n$ is a sample if i.i.d. observations, and the variance of the asymptotic normal distribution becomes the variance of X,

$$\lim_{\epsilon \to 1} \sigma_\epsilon = DX.$$

Therefore, we obtain as a special case the classical CLT

$$\frac{\sqrt{n}}{\sqrt{DX}}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - EX\right) \xrightarrow{w} N(0,1).$$

3 Monte Carlo experiments

In this section, our goal is to investigate the effect of the tail behavior on the rate of convergence in (1). We are also interested in the question if tail truncation improves the convergence and by how much. Generally, the tail truncation method consists in "replacing" the tails of X with the tails of a thin-tailed distribution "far away" from the center of the distribution of X, for example beyond the 0.1% and 99.9% quantiles. The tail truncation method has applications in finance for modeling the distribution of stock returns, a practical reason being that stock exchanges close if a severe market crash occurs. This method also has application in derivatives pricing with a heavy-tailed distributional assumption for the return of the underlying, see Rachev et al. (2005) and the references therein.

In the following sections, we start with Student's t distribution and investigate the convergence rate in the limit relation (1) as degrees of freedom increase. We address the same questions with a truncated Student's t distribution in which the truncation is done in the simplest possible way — we set the the values of the random variable which are beyond the 0.1% and 99.9% quantiles to be equal to the corresponding quantiles. As a result, small point masses appear at the 0.1% and 99.9% quantiles. We also focus on the class of stable distributions and truncated stable distributions in which the same truncation technique is adopted as in the case of Student's t distribution.

3.1 The effect of tail thickness

The impact of the tail behavior on the rate of convergence in Theorem 1 is first studied when X has Student's t distribution, $X \in t(\nu)$, with $\nu \geq 3$. We need the condition on the degrees of freedom in order for the random variable to have finite variance. Taking advantage of the expression for the density,

$$f_\nu(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \ x \in \mathbb{R},$$

it is possible to compute explicitly the variance in equation (2). In fact, for this purpose the expression in (4) is more appropriate. As a first step, we calculate the two conditional expectations.

$$\begin{aligned} E(X|X \leq q_\epsilon) &= \frac{1}{\epsilon} \int_{-\infty}^{q_\epsilon} x f_X(x) dx \\ &= \frac{1}{\epsilon} \int_{-\infty}^{q_\epsilon} x \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} dx \\ &= \frac{1}{\epsilon} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \frac{\nu}{2} \int_{-\infty}^{q_\epsilon} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} d\left(1 + \frac{x^2}{\nu}\right) \\ &= -\frac{1}{\epsilon} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{\sqrt{\nu}}{(\nu-1)\sqrt{\pi}} \left(1 + \frac{q_\epsilon^2}{\nu}\right)^{\frac{1-\nu}{2}}, \ \text{if } \nu > 1. \end{aligned} \tag{5}$$

$$\begin{aligned} E(X^2|X \leq q_\epsilon) &= \frac{1}{\epsilon} \int_{-\infty}^{q_\epsilon} x^2 f_X(x) dx \\ &= \frac{1}{\epsilon} \int_{-\infty}^{q_\epsilon} x^2 \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} dx \\ &= \frac{1}{\epsilon} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \frac{\nu}{1-\nu} \int_{-\infty}^{q_\epsilon} x d\left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}+1} \\ &= q_\epsilon E(X|X \leq q_\epsilon) + \frac{\nu}{\epsilon(\nu-2)} F_{\nu-2}\left(q_\epsilon \sqrt{\frac{\nu-2}{\nu}}\right), \ \text{if } \nu > 2. \end{aligned} \tag{6}$$

where the last equality follows by integration by parts and $F_\nu(x)$ is the c.d.f. of Student's t distribution with ν degrees of freedom. Plugging these expressions in (4), we obtain the expression for the variance σ_ϵ^2.

ν	$\epsilon = 0.01$	$\epsilon = 0.05$
3	70000	17000
4	60000	9000
5	50000	7000
6	23000	4500
7	14000	4200
8	13000	4100
9	12000	4000
10	12000	3900
15	11000	3850
25	10000	3800
50	10000	3750
∞	10000	3300

Table 1: The number of observations sufficient to accept the normal distribution as an approximate model for different values of ν and ϵ.

Note that, besides an equation for σ_ϵ^2, we can explicitly calculate the AVaR of X since in the case of Student's t distribution we can express AVaR as a conditional expectation,

$$AVaR_\epsilon(X) = -E(X|X \leq q_\epsilon).$$

Having an expression for the variance allows us to use the test of Kolmogorov and address the question oh how many simulations are needed in order to accept the hypothesis that the distribution of the random variable in the left-hand side of the limit relation (1),

$$\frac{\sqrt{n}}{\sigma_\epsilon}\left(\widehat{AVaR}_\epsilon(X) - AVaR_\epsilon(X)\right), \tag{7}$$

is standard normal. If we accept the null hypothesis for a given value of n, then the standard normal distribution can be used as an approximate model and we can calculate not only confidence intervals but also other characteristics based on it.

Table 1 shows the values of n sufficient to accept the null hypothesis in the test of Kolmogorov for different degrees of freedom and tail probabilities. We chose $\epsilon = 0.01$ and $\epsilon = 0.05$ since these values are frequently used in financial industry in value-at-risk estimation. The numbers in the table are calculated by generating independently 2000 samples of a given size and then from each sample (7) is estimated. In effect, we obtain 2000 observations from the distribution of (7).

In line with intuition, the numbers Table 1 indicate that when the tail

	$n = 250$		$n = 500$		$n = 1000$		$n = 5000$		$n = 10000$	
ν	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$
3	-1.110	2.011	-1.257	2.173	-1.352	2.202	-1.633	2.037	-1.664	2.007
4	-1.337	2.144	-1.442	2.229	-1.543	2.082	-1.744	2.230	-1.756	2.176
5	-1.441	2.153	-1.529	2.224	-1.728	2.190	-1.843	2.060	-1.807	2.009
6	-1.522	2.134	-1.618	2.033	-1.701	2.115	-1.848	1.987	-1.955	1.982
7	-1.627	2.050	-1.668	1.975	-1.827	2.043	-1.841	2.048	-1.913	2.014
8	-1.655	2.028	-1.760	2.145	-1.836	2.032	-1.898	2.034	-1.866	1.939
9	-1.720	1.938	-1.753	2.146	-1.798	2.075	-1.866	2.005	-1.905	2.007
10	-1.747	1.925	-1.809	1.980	-1.762	2.078	-1.822	1.950	-1.962	2.000
15	-1.813	1.751	-1.848	1.896	-1.891	1.956	-1.969	1.941	-1.968	1.873
25	-1.848	1.760	-1.933	2.028	-1.897	1.950	-1.939	1.957	-1.899	1.923
50	-1.898	1.948	-1.962	1.900	-1.971	1.973	-1.961	1.914	-1.895	1.948
∞	-1.921	1.761	-1.976	1.920	-1.964	1.822	-1.869	1.907	-2.004	1.937

Table 2: The 95% confidence bounds generated from 2000 simulations from the distribution of (7) with $\epsilon = 0.01$. The corresponding quantiles of $N(0,1)$ are $q_{2.5\%} = -1.96$ and $q_{2.5\%} = 1.96$.

is heavier, we need a larger sample in order for the asymptotic law to be sufficiently close to the distribution of (7) in terms of the Kolmogorov metric. Another expected conclusion is that as the tail probability increases, a smaller sample turns out to be sufficient.

In Table 2, we calculated the 95% confidence interval for AVaR when the sample size changes from 250 to 10000 observations. We generated 2000 independent samples and then computed the quantity in equation (7). Thus, the 95% confidence intervals are obtained from 2000 observations of the random variable in (7). As n increases, the two quantiles approach the corresponding quantiles of the standard normal distribution. Note that the largest $n = 10000$ is generally below the sample sizes for $\epsilon = 0.01$ given in Table 1. Nevertheless, the relative discrepancies between the quantiles given in Table 2 and the corresponding standard normal distribution quantiles are less than 5% for $\nu \geq 6$.[3] The relative discrepancies between the quantiles given in Table 3 the corresponding standard normal distribution quantiles for $n = 10000$ have the same magnitude. However, in this case $n = 10000$ is well above the sample sizes given in Table 1 for $\epsilon = 0.05$. As a result, we can conclude that even smaller samples than the ones given in Table 1 can lead to 95% confidence intervals obtained via resampling from (7) being close to the

[3] If we generate a sample of 2000 observations from the standard normal distribution, a relative deviation below 6% between the estimated quantile $q_{2.5\%}$ and the corresponding standard normal quantile happens with about 95% probability, and below 7.7% with about 99% probability.

	$n=250$		$n=500$		$n=1000$		$n=5000$		$n=10000$	
ν	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$
3	-1.422	2.110	-1.543	2.016	-1.549	1.981	-1.725	1.947	-1.883	1.987
4	-1.647	2.169	-1.737	2.235	-1.787	2.171	-1.900	2.226	-1.849	2.115
5	-1.749	2.081	-1.811	2.096	-1.757	2.148	-1.868	2.015	-1.937	2.100
6	-1.810	2.071	-1.896	2.030	-1.921	1.941	-1.958	1.998	-1.886	2.032
7	-1.786	2.215	-1.824	1.990	-1.809	2.086	-1.986	2.030	-1.916	2.015
8	-1.932	2.131	-1.870	2.058	-1.755	2.090	-1.937	2.014	-1.915	1.952
9	-1.848	2.139	-1.884	2.081	-1.930	2.023	-1.995	1.964	-1.863	2.048
10	-1.906	2.103	-2.021	1.966	-1.839	2.087	-2.009	1.930	-1.989	1.995
15	-1.797	1.905	-1.929	2.056	-1.944	1.952	-1.924	1.973	-1.947	1.979
25	-1.958	1.950	-1.994	1.956	-1.939	1.968	-2.085	1.993	-1.894	1.944
50	-1.986	1.927	-1.980	1.823	-1.962	1.883	-1.911	1.969	-2.002	1.935
∞	-2.013	1.828	-1.953	1.869	-1.975	1.893	-2.034	1.958	-1.903	1.944

Table 3: The 95% confidence bounds generated from 2000 simulations from the distribution of (7) with $\epsilon = 0.05$. The corresponding quantiles of $N(0,1)$ are $q_{2.5\%} = -1.96$ and $q_{2.5\%} = 1.96$.

corresponding 95% confidence interval obtained from the limit distribution even though the Kolmogorov test fails for such samples. For instance, the relative deviation between the quantiles given in Table 2 for $n = 5000$ and the corresponding standard normal distribution quantiles are below 7% for $n \geq 6$, which is a small deviation for all practical purposes.

As a result of this analysis, we can conclude that for the purposes of building confidence intervals for $AVaR_\epsilon(X)$ when $X \in t(\nu)$, with $\nu \geq 6$ and $\epsilon = 0.01, 0.05$, we can safely employ the asymptotic law when the sample size we use for AVaR estimation contains more than 5000 observations. If Student's t distribution is fitted on daily stock-returns time series, such values for ν are very common.

Figure 1 illustrates the differences in the convergence rate when X has Student's t distribution with $\nu = 3$, which corresponds to heavier tails, and $\nu = 10$. Since high degrees of freedom imply more light tails, smaller samples are sufficient for the density of (7) to be closer to the standard normal density.

3.2 The effect of tail truncation

The stochastic stability of sample AVaR increases dramatically after tail truncation. In this section, we repeat the calculations from the previous section but when X has Student's t distribution with the tails truncated at $q_{0.1\%}$ and $q_{99.9\%}$ quantiles. The random variable Y is said to have a truncated distribution at these quantiles if it has the representation

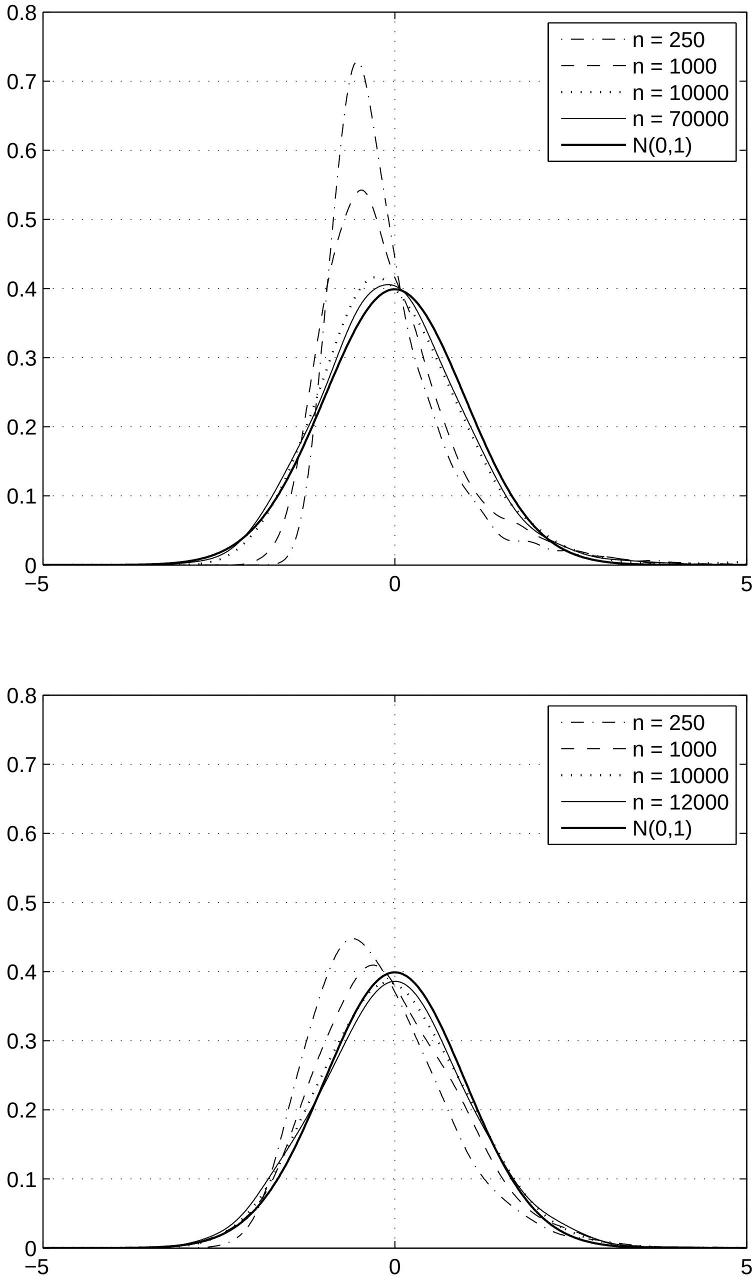

Figure 1: The density of (7) approaching the $N(0, 1)$ density as the sample size increases with $\nu = 3$ (top) and $\nu = 10$ (bottom).

ν	$\epsilon = 0.01$	$\epsilon = 0.05$
3	12000	4000
4	11500	3600
5	11000	3300
6	11000	3200
7	10500	3100
8	10000	3000
9	10000	3000
10	10000	3000
15	10000	2950
25	10000	2900
50	10000	2900
∞	10000	2900

Table 4: The number of observations sufficient to accept the normal distribution as an approximate model for different values of ν and ϵ.

$$Y = XI\{q_{0.1\%} \leq X \leq q_{99.9\%}\} + q_{0.1\%}I\{X < q_{0.1\%}\} + q_{99.9\%}I\{X > q_{99.9\%}\}$$

in which $X \in t(\nu)$, $I\{A\}$ denotes the indicator of the event A, and $q_{0.1\%}$, $q_{99.9\%}$ are the corresponding quantiles of X. The tail truncation introduces small point masses at the two quantile levels.

The two conditional expectations in (4) can be related to the corresponding conditional expectations of X. In the following, we assume that the tail probability ϵ is larger from the tail probability of the left truncation point, $\epsilon > 0.001$. Under this assumption, the ϵ-quantile of X is the same as the ϵ-quantile of Y.

$$E(Y|Y \leq q_\epsilon) = E(X|X \leq q_\epsilon) - \frac{0.001}{\epsilon}E(X|X \leq q_{0.1\%}) + \frac{0.001q_\epsilon}{\epsilon}$$

$$E(Y^2|Y \leq q_\epsilon) = E(X^2|X \leq q_\epsilon) - \frac{0.001}{\epsilon}E(X^2|X \leq q_{0.1\%}) + \frac{0.001q_\epsilon^2}{\epsilon}$$

in which the conditional expectations of X can be computed according to formulae (5) and (6). Plugging the expressions for the conditional expectations of Y in the expression for σ_ϵ^2, we obtain the variance of the asymptotic distribution. Furthermore, the tail truncation does not break the link between AVaR and the conditional expectation, therefore

$$AVaR_\epsilon(Y) = -E(Y|Y \leq q_\epsilon).$$

	$n = 250$		$n = 500$		$n = 1000$		$n = 5000$		$n = 10000$	
ν	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$
3	-1.723	1.699	-1.847	1.932	-1.850	1.958	-1.966	1.921	-1.860	1.936
4	-1.759	1.694	-1.863	1.819	-1.903	1.860	-1.989	1.942	-1.964	1.886
5	-1.808	1.536	-1.884	1.871	-1.926	1.932	-1.961	1.964	-1.782	2.066
6	-1.947	1.565	-1.937	1.759	-2.002	1.734	-2.057	1.946	-1.981	1.958
7	-1.960	1.524	-1.960	1.666	-1.965	1.844	-2.101	1.932	-1.981	1.927
8	-2.002	1.567	-2.015	1.693	-1.903	1.802	-1.952	1.856	-1.917	1.928
9	-1.963	1.552	-2.030	1.748	-2.106	1.779	-1.965	2.026	-1.932	1.938
10	-2.003	1.596	-2.119	1.709	-2.034	1.850	-1.925	1.813	-1.990	1.952
15	-2.090	1.485	-2.159	1.650	-2.065	1.786	-1.983	1.847	-2.035	1.855
25	-2.183	1.502	-2.084	1.578	-2.093	1.747	-2.016	1.806	-1.954	1.877
50	-2.272	1.509	-2.089	1.632	-2.042	1.726	-1.938	1.914	-2.056	1.970

Table 5: The 95% confidence bounds generated from 2000 simulations from the distribution of (8) with $\epsilon = 0.01$. The corresponding quantiles of $N(0,1)$ are $q_{2.5\%} = -1.96$ and $q_{2.5\%} = 1.96$.

In the following, we investigate the convergence rate of

$$\frac{\sqrt{n}}{\sigma_\epsilon}\left(\widehat{AVaR}_\epsilon(Y) - AVaR_\epsilon(Y)\right), \tag{8}$$

for different degrees of freedom to the standard normal distribution and we compare the results to the ones in the previous section.

Table 4 is the counterpart of Table 1 for the truncated distribution. It is impressive how the sample size sufficient to accept the null hypothesis in the Kolmogorov test decreases after tail truncation. The most dramatic change is in the case $\nu = 3$. Now we need only 12000 observations compared to 70000 in the non-truncated case.

Tables 5 and 6 are the counterparts of Tables 2 and 3. The relative deviation of the quantiles $q_{2.5\%}$ and $q_{97.5\%}$ of the random variable in (8) from those of the standard normal distribution are below 7% for all degrees of freedom and $n = 10000$, and, with a few exceptions, for $n = 5000$. Compare Figure 2 and the top plot in Figure 1 for an illustration of the improvement in the convergence rate. These results indicate that the asymptotic distribution can be used to obtain a 95% confidence bound for the sample AVaR for all degrees of freedom if the sample size contains more than 5000 observations.

3.3 Infinite variance distributions

A critical assumption behind the limit result in Theorem 1 is the finite variance of X. To be more precise, the condition of finite variance can be loosened

	$n = 250$		$n = 500$		$n = 1000$		$n = 5000$		$n = 10000$	
ν	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$
3	-1.815	2.116	-1.866	2.041	-1.939	2.018	-1.944	1.975	-2.045	1.874
4	-1.756	2.150	-1.811	2.073	-2.052	2.060	-1.923	1.973	-1.922	1.854
5	-1.820	1.954	-1.971	2.032	-1.916	2.036	-1.826	1.960	-1.941	1.883
6	-1.899	2.089	-1.981	2.036	-2.012	2.012	-1.955	1.933	-1.921	2.011
7	-2.001	2.032	-1.921	1.997	-1.949	1.980	-1.980	1.936	-2.016	1.915
8	-1.888	1.995	-1.922	2.050	-1.907	1.917	-1.942	1.911	-1.910	1.903
9	-2.017	2.003	-1.892	1.918	-1.899	2.017	-1.931	2.001	-2.009	1.967
10	-1.928	1.814	-1.992	1.960	-1.870	1.949	-1.845	2.076	-1.992	1.898
15	-2.059	1.983	-2.020	2.007	-1.961	1.922	-1.953	1.870	-1.936	1.874
25	-1.999	1.854	-2.038	1.945	-1.889	2.028	-2.031	1.916	-1.975	1.890
50	-1.960	1.898	-2.028	1.898	-1.947	1.906	-2.015	2.002	-1.959	1.911

Table 6: The 95% confidence bounds generated from 2000 simulations from the distribution of (8) with $\epsilon = 0.05$. The corresponding quantiles of $N(0,1)$ are $q_{2.5\%} = -1.96$ and $q_{2.5\%} = 1.96$.

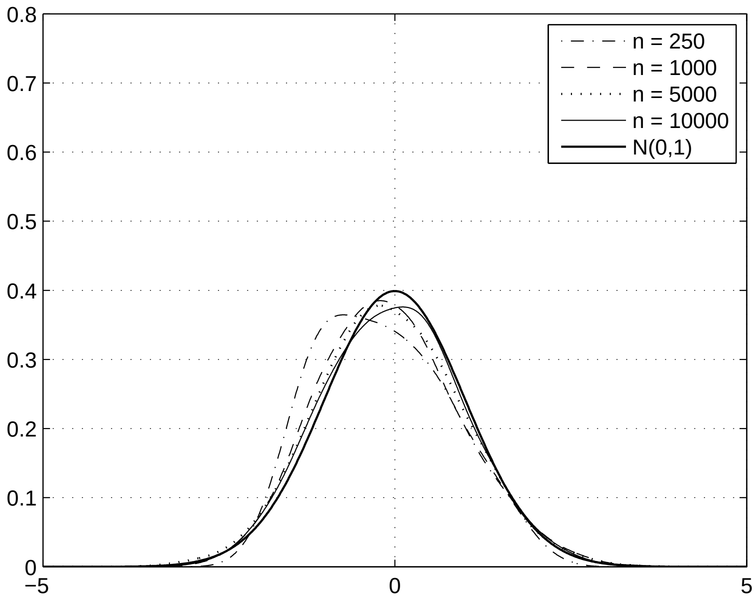

Figure 2: The density of (8) approaching the $N(0,1)$ density as the sample size increases with $\nu = 3$ and $\epsilon = 0.01$.

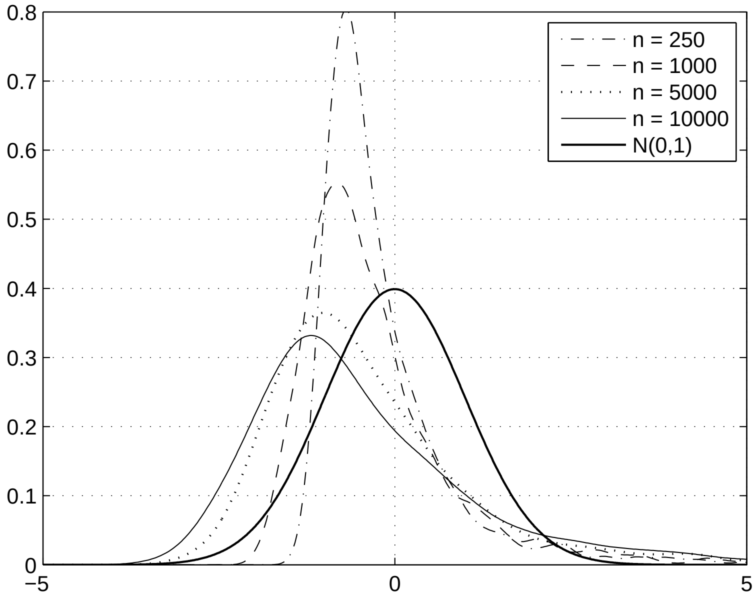

Figure 3: Lack of convergence, X has a stable distribution with $X \in S_{1.5}(1, 0, 0)$ and $\epsilon = 0.05$.

to finite downside semi-variance,

$$D \max(-X, 0) < \infty,$$

because it is the behavior of the left tail which is important. As a consequence, the sample AVaR of distributions with infinite variance, but finite downside semi-variance, may still follow Theorem 1.

However, there are infinite variance distributions for which

$$D \max(-X, 0) = \infty$$

and, therefore, the limit result in Theorem 1 does not hold for them. Such is the class of stable distributions which arises from generalizations of the Central Limit Theorem and has been proposed as a model for stock return distributions, see Rachev and Mittnik (2000).

Stable distributions are introduced by their characteristic functions. X is said to have a stable distribution if its characteristic function is

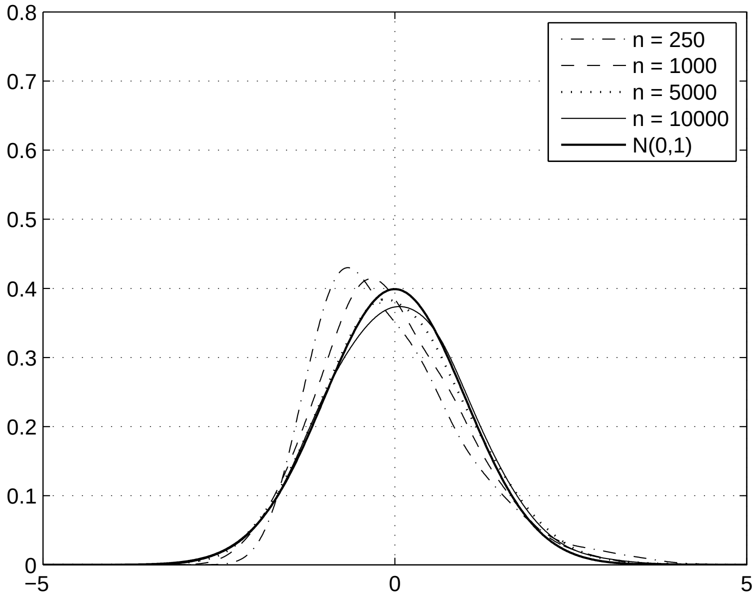

Figure 4: After tail truncation at $q_{0.1\%}$ and $q_{99.9\%}$, there is a fast convergence to $N(0,1)$, $\alpha = 1.5$ and $\epsilon = 0.05$.

$$\varphi(t) = Ee^{itX} = \begin{cases} \exp\{-\sigma^\alpha |t|^\alpha (1 - i\beta \frac{t}{|t|} \tan(\frac{\pi\alpha}{2})) + i\mu t\}, & \alpha \neq 1 \\ \exp\{-\sigma |t| (1 + i\beta \frac{2}{\pi} \frac{t}{|t|} \ln(|t|)) + i\mu t\}, & \alpha = 1 \end{cases}$$

Except for a couple of representatives, generally no closed-form expressions for their densities and c.d.f.s are known. If $\alpha < 2$, then X has infinite variance. If $1 < \alpha \leq 2$, then X has finite mean and the AVaR of X can be calculated. In our calculations, we will use the semi-analytic formula in Stoyanov et al. (2006).

Even though we know that Theorem 1 does not hold for a stable distribution with $\alpha < 2$, we simulate 2000 draws from the random variable in equation (7) in which σ_ϵ is estimated from a generated sample by estimating the corresponding conditional moments. In theory these the second conditional moment explodes but for any finite sample its estimate is a finite number. Our goal is to see what happens when Theorem 1 does not hold. Figure 3 illustrates such a divergent case in which $\alpha = 1.5$ and $\epsilon = 0.05$. The lack of convergence is quite obvious.

Stable distributions with $\alpha < 2$ in combination with a tail truncation method have been proposed as a model for the returns of the underlying in derivatives pricing. It is interesting to see how much the simple truncation technique we applied in the previous section can change Figure 3. With its tails truncated according to our simple method, the random variable becomes with a bounded support and, therefore, it has finite variance. As a consequence, Theorem 1 holds. Figure 4 illustrates this change. We observe a quick convergence rate, similar to the one illustrated in Figure 2 for Student's t distribution.

4 Conclusion

In this paper, we study the asymptotic distribution of sample AVaR. Under certain regularity conditions, we prove a limit theorem in which the limiting distribution is the normal distribution. We study how the convergence rate in the limit theorem is influenced by the tail behavior of the random variable. An expected result is that, other things equal, more observations are needed when the tail is heavier. We find out that a simple tail truncation method improves dramatically the convergence rate. As a consequence, the asymptotic distribution is reliable for confidence interval calculations when the number of simulations is more than 5000 if the random variable has a truncated Student's t distribution.

We also consider an infinite variance case in which the random variable as a stable distribution with finite mean. We illustrate the lack of convergence and demonstrate the improvement due to tail truncation at high quantiles.

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