Technical Appendix

Lecture 9: Benchmark tracking problems

Prof. Dr. Svetlozar Rachev

Institute for Statistics and Mathematical Economics
University of Karlsruhe

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The material is based on the text-book:

Svetlozar T. Rachev, Stoyan Stoyanov, and Frank J. Fabozzi

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Prof. Svetlozar (Zari) T. Rachev
Chair of Econometrics, Statistics
and Mathematical Finance
School of Economics and Business Engineering
University of Karlsruhe
Kollegium am Schloss, Bau II, 20.12, R210
Postfach 6980, D-76128, Karlsruhe, Germany
Tel. +49-721-608-7535, +49-721-608-2042(s)
Fax: +49-721-608-3811
http://www.statistik.uni-karslruhe.de

Deviation measures and r.d. metrics

- If μ is a translation invariant and positively homogeneous of degree 1 probability metric, then the functional D_μ is a symmetric deviation measure.
- If D is a symmetric deviation measure, then the functional μ_D is a translation invariant and positively homogeneous of degree 1 probability semimetric.
- Symmetry properties of μ and D influence only the symmetry properties of D_μ and μ_D, respectively.
- Relaxing the assumption that μ is symmetric (Property 2) results in D_μ being asymmetric as well.
- If μ is a translation invariant and positively homogeneous of degree 1 r.d. metric, then D_μ is a general deviation measure.
- If D is a general deviation measure, then μ_D is a translation invariant and positively homogeneous of degree 1 r.d. metric.

Remarks on the axioms

- Recall that a probability semi-metric which does not satisfy the symmetry axiom SYM is called probability quasi-semi-metric.

How we can modify a probability metric so that it becomes better suited for the benchmark-tracking problem.

- Let us choose two classical examples of compound probability metrics — the average compound metric

$$\mathcal{L}_p(X, Y) = (E|X - Y|^p)^{1/p}, \; p \geq 1$$

and the Birnbaum-Orlicz compound metric

$$\mathbf{\Theta}_p(X, Y) = \left[\int_{-\infty}^{\infty} (\tau(t; X, Y))^p dt \right]^{1/p}, \; p \geq 1$$

where $\tau(t; X, Y) = P(X \leq t < Y) + P(Y \leq t < X)$.

- Both $\mathcal{L}_p(X, Y)$ and $\mathbf{\Theta}_p(X, Y)$ are ideal because they satisfy the positive homogeinity property and the weak regularity property.

Remarks on the axioms

- The average compound metric satisfies all properties of relative deviation metrics but it is symmetric, a property we would like to break.
- One possible way is to replace the absolute value by the max function, obtaining the asymmetric version

$$\mathcal{L}_p^*(X,Y) = (E(\max(Y-X,0))^p)^{1/p},\ p \geq 1. \quad (1)$$

- In Stoyanov, Rachev, Ortobelli and Fabozzi (2007) we show that $\mathcal{L}_p^*(X,Y)$ is an ideal quasi-semi-metric; that is, using the max function instead of the absolute value breaks only the symmetry axiom SYM.

Remarks on the axioms

What is the intuition behind removing the absolute value and considering the max function?

- Suppose that the r.v. X stands for the return of the portfolio and Y denotes the return of the benchmark. The difference $Y - X$ can be interpreted as the portfolio loss relative to the benchmark, or the portfolio underperformance.
- If in a given state of the world, $\omega \in \Omega$, the difference is positive, $Y(\omega) - X(\omega) > 0$, then in this state of the world the portfolio is underperforming the benchmark.

Remarks on the axioms

- The expectation

$$\mathcal{L}_1^*(X, Y) = E \max(Y - X, 0)$$

  measures the average portfolio underperformance. When we minimize $\mathcal{L}_1^*$ in the optimization problem, we are actually minimizing the average portfolio underperformance.
- The same is idea behind the general case $\mathcal{L}_p^*(X, Y)$. There is additional flexibility in that the power $p \geq 1$ allows increasing the importance of the larger losses by increasing p.

Remarks on the axioms

- The absolute difference $|X - Y|$ in the classical probability metric $\mathcal{L}_1(X, Y)$ is either underperformance or outperformance of the benchmark depending on whether the difference $Y(\omega) - X(\omega)$ is positive or negative in a given state of the world $\omega \in \Omega$.
- The absolute difference can be decomposed into an underperformance and an outperformance term

$$|X(\omega) - Y(\omega)| = \max(X(\omega) - Y(\omega), 0) + \max(Y(\omega) - X(\omega), 0).$$

- If the first summand is positive, then we have outperformance and if the second summand is positive we have underperformance in the corresponding state of the world $\omega \in \Omega$.
- If we minimize $\mathcal{L}_1(X, Y)$ in the benchmark tracking problem, then we minimize *simultaneously* both the portfolio outperformance and underperformance.

$\Rightarrow$ A similar conclusion holds for the general case $\mathcal{L}_p(X, Y)$.

Remarks on the axioms

- The same idea, but implemented in a different way, stays behind the asymmetric version of the Birnbaum-Orlicz metric

$$\mathbf{\Theta}_p^*(X, Y) = \left[\int_{-\infty}^{\infty} (\tau^*(t; X, Y))^p dt \right]^{1/p}, \ p \geq 1 \qquad (2)$$

where $\tau^*(t; X, Y) = P(X \leq t < Y)$. In Stoyanov, Rachev, Ortobelli and Fabozzi (2007) we show that (2) is an ideal quasi-semi-metric.

- That is, considering only the first summand of the function $\tau(t; X, Y)$ from the Birnbaum-Orlics compound metric breaks the SYM axiom only.

Remarks on the axioms

Interpreting the integrand — the $\tau^*(t; X, Y)$ function:

- Just as in the case of the asymmetric version of the average compound metric, suppose that the r.v. X represents the return of the portfolio and Y represents the benchmark return.
- For a fixed value of the argument t, which we interpret as a threshold, the function τ^* calculates the probability of the event that the portfolio return is below the threshold t and, simultaneously, the benchmark return is above the threshold t,

$$\tau^*(t; X, Y) = P(X \leq t < Y) = P(\{X \leq t\} \cap \{t < Y\}).$$

- The function τ^* calculates the probability that the portfolio return is below the benchmark return with respect to the threshold t.

Remarks on the axioms

- As a result, we can interpret $\boldsymbol{\Theta}_p^*(X, Y)$ as a measure of the probability that portfolio loses more than the benchmark.
- In the benchmark-tracking problem, by minimizing $\boldsymbol{\Theta}_p^*(X, Y)$, we are indirectly minimizing the probability of the portfolio losing more than the benchmark.
- Interestingly, the special case $p = 1$,

$$\boldsymbol{\Theta}_1^*(X, Y) = \int_{-\infty}^{\infty} \tau^*(t; X, Y) dt$$

allows for a very concrete interpretation.
- $\boldsymbol{\Theta}_1^*(X, Y)$ is exactly equal to the average underperformance; that is $\boldsymbol{\Theta}_1^*(X, Y) = \mathcal{L}_1^*(X, Y)$. This holds because $\boldsymbol{\Theta}_1^*(X, Y)$ is just an alternative way of writing down the integral behind the expectation in $\mathcal{L}_1^*(X, Y)$.

Minimal r.d. metrics

- Any probability metric is defined on a pair of random variables (X, Y).
- Depending on the implied equivalence in property ID, we distinguish between three classes of metrics — primary, simple and compound:
  - The *primary metrics* imply the weakest form of sameness, only up to equality of certain characteristics.
  - The *simple metrics* have stronger implications. It is only if the distribution functions of the random variables agree completely that the measured distance between them becomes zero.
  - The *compound metrics* imply the strongest possible identity — in almost sure sense.

Minimal r.d. metrics

There are links between the corresponding classes:

- By including more and more characteristics we obtain primary metrics which essentially require that the distribution functions of the random variables should coincide; that is, they turn into simple metrics.
- By minimizing any compound metric on all possible dependencies between the two random variables we obtain a metric which actually depends only on the distribution functions and is, therefore, simple. This is the construction of the minimal metric which is defined by

$$\hat{\mu}(X, Y) = \inf\{\mu(\widetilde{X}, \widetilde{Y}) : \widetilde{X} \stackrel{d}{=} X, \widetilde{Y} \stackrel{d}{=} Y\}$$

Minimal r.d. metrics

- It is possible to construct the minimal r.d. metrics in the same manner as minimal probability metrics. The approach can be used to construct nontrivial examples of simple r.d. metrics such as **(7)** and **(8)** from the lecture.
- It is possible to show that, if μ is a functional satisfying properties ID or $\widetilde{\mathrm{ID}}$, TI or $\widetilde{\mathrm{TI}}$, then $\hat{\mu}$ also satisfies ID or $\widetilde{\mathrm{ID}}$, TI or $\widetilde{\mathrm{TI}}$.
- Omitting the symmetry property results only in asymmetry in the minimal functional $\hat{\mu}$ and influences nothing else. These are, basically, the results in the proof that $\hat{\mu}$ is a probability (semi)distance.
- It is easy to check that if positive homogeneity holds for μ, then the same property holds for $\hat{\mu}$ as well. The same holds for the weak regularity.

Minimal r.d. metrics

- The construction of the minimal r.d. metric, just as the minimal probability metric, is an important tool because some of the properties above are easy to check for a compound metric and difficult to check for a simple metric.
- For example, this is the case with the weak regularity property. Therefore, starting with a compound r.d. metric, we are sure that the minimal functional corresponding to it is a simple r.d. metric.
- Sometimes, it is possible to calculate explicitly the minimal functional. This can be done either through the Cambanis-Simons-Stout theorem or through the Frechet-Hoffding inequality.

Minimal r.d. metrics

- Now we will show how the Cambanis-Simons-Stout result is applied to the functional

$$\mathcal{L}_p^*(X, Y) = (E(\max(Y - X, 0))^p)^{1/p}, \; p \geq 1.$$

- It is easy to check that $\widetilde{\mathrm{ID}}$, TI, $\widetilde{\mathrm{RE}}$, and Property 4 hold for $\mathcal{L}_p^*(X, Y)$.
- We identify the function ϕ, $\phi(x, y) = (\max(x - y, 0))^p$. Clearly $\phi(x, x) = 0$ and ϕ is quasi-antitone because $f(x) = (\max(x, 0))^p, \; p \geq 1$ is a non-negaive, convex function.
- The Cambanis-Simons-Stout theorem applies and, therefore, the minimal functional is given by

$$\ell_p^*(X, Y) = \hat{\mathcal{L}}_p^*(X, Y) = \left[\int_0^1 (\max(F_Y^{-1}(t) - F_X^{-1}(t), 0))^p dt\right]^{1/p}$$

which is equation **(8)** in the lecture.

Minimal r.d. metrics

- There is another method of obtaining explicit forms of minimal functionals via the celebrated Frechet-Hoeffding inequality between distribution functions defined in **(26)** in the appendix to Lecture 3.
- We show how this inequality is applied to the problem of finding the minimal r.d. metric of the Birnbaum-Orlicz quasi-semi-metric defined in (2) by taking advantage of the upper bound.

Minimal r.d. metrics

- Consider the following representation of the τ^* function defined in (2),

$$\begin{aligned}\tau^*(t; X, Y) &= P(X \leq t, Y < t) \\ &= P(X \leq t) - P(X \leq t, Y \leq t).\end{aligned}$$

- This representation is correct because by summing $P(X \leq t, Y > t)$ and $P(X \leq t, Y \leq t)$, the influence of the random variable Y is cancelled out.
- By replacing the joint probability by the upper bound from the Frechet-Hoeffding inequality, we obtain

$$\begin{aligned}\tau^*(t; X, Y) &\geq F_X(t) - \min(F_X(t), F_Y(t)) \\ &= \max(F_X(t) - F_Y(t), 0).\end{aligned}$$

Minimal r.d. metrics

- Raising both sides of the above inequality to the power $p \geq 1$ and integrating over all values of t does not change the inequality.
- We obtain

$$\left[\int_{-\infty}^{\infty} (\max(F_X(t) - F_Y(t), 0))^p dt\right]^{1/p} \leq \mathbf{\Theta}_p^*(X, Y)$$

  which gives, essentially, the corresponding minimal r.d. metric.
- The left side of the inequality coincides with **(7)** from the lecture, $\theta_p^*(X, Y) = \hat{\mathbf{\Theta}}_p^*(X, Y)$.

Limit cases of $\mathcal{L}_p^*(X, Y)$ and $\Theta_p^*(X, Y)$

There are several limit cases of the two r.d. metrics which help better understand their behaviour. We will consider the most intuitive ones.

- In line with the setting of the benchmark-tracking problem, in the interpretations we assume that X represents portfolio return and Y represents the benchmark return.
- There are two ways to obtain limit representatives — if we let p approach infinity, or zero.
- We defined both r.d. metrics for $p \geq 1$ and we will slightly change the definitions so that we can see what is going on as $p \to 0$.

Limit cases of $\mathcal{L}^*_p(X, Y)$ and $\Theta^*_p(X, Y)$

- The slightly extended definitions are,

$$\mathcal{L}^*_p(X, Y) = (E(\max(Y - X, 0))^p)^{1/\min(1,1/p)}, \; p \geq 0 \qquad (3)$$

and

$$\Theta^*_p(X, Y) = \left[\int_{-\infty}^{\infty} (\tau^*(t; X, Y))^p dt\right]^{1/\min(1,1/p)}, \; p \geq 0. \qquad (4)$$

$\Rightarrow$ The change affects the case $p \in [0, 1)$ and if $p \geq 1$, then we obtain the previous definitions.

Limit cases of $\mathcal{L}_p^*(X,Y)$ and $\Theta_p^*(X,Y)$

- As $p \to \infty$, the r.d. metric $\mathcal{L}_p^*(X,Y)$ approaches $\mathcal{L}_\infty^*(X,Y)$ defined as
$$\mathcal{L}_\infty^*(X,Y) = \inf\{\epsilon > 0 : P(\max(Y - X, 0) > \epsilon) = 0\}$$
- This limit case can be interpreted in the following way. $\mathcal{L}_\infty^*(X,Y)$ calculates the smallest threshold so that the portfolio loss relative to the benchmark is larger than this threshold with zero probability.

$\Rightarrow$ Note that this quasi-semi-metric is entirely focused on the very extreme loss.

Limit cases of $\mathcal{L}^*_p(X, Y)$ and $\Theta^*_p(X, Y)$

- In the other direction, if $p \to 0$, the r.d. metric $\mathcal{L}^*_p(X, Y)$ approaches $\mathcal{L}^*_0(X, Y)$ where

$$\begin{aligned}\mathcal{L}^*_0(X, Y) &= EI\{\omega : \max(Y(\omega) - X(\omega), 0) \neq 0\}\\ &= P(Y > X).\end{aligned}$$

- The notation $I\{\omega \in A\}$ stands for the indicator function of the event A, i.e. if ω happens to be in A, then $I\{\omega \in A\} = 1$ and otherwise it is equal to zero.
- This result is self-explanatory, $\mathcal{L}^*_0(X, Y)$ calculates the probability of the event the portfolio to lose relative to the benchmark.

Limit cases of $\mathcal{L}_p^*(X, Y)$ and $\mathbf{\Theta}_p^*(X, Y)$

- Concerning the Birnbaum-Orlicz quasi-semi-metric given by (4), there is an interesting limit case as $p \to \infty$,

$$\mathbf{\Theta}_\infty^*(X, Y) = \sup_{t \in R} P(X \leq t < Y).$$

- Let us briefly look at the properties of the function $\tau^*(t; X, Y) = P(X \leq t < Y)$ in order to see what this limit case calculates.
- As t decreases to $-\infty$, the sets $\{\omega : X(\omega) \leq t\}$ become progressively smaller and at the limit they approach the empty set, $\lim_{t \to \infty} \{\omega : X(\omega) \leq t\} = \emptyset$.
- The same conclusion is valid for the sets $\{\omega : Y(\omega) > t\}$ as t increases to infinity.

Limit cases of $\mathcal{L}_p^*(X, Y)$ and $\Theta_p^*(X, Y)$

- Since the function $\tau^*(t; X, Y)$ is, essentially, the probability of the intersection of these two events, it follows that $\tau^*(t; X, Y)$ decays to zero as t decreases or increases unboundedly,

$$\lim_{t \to -\infty} \tau^*(t; X, Y) = 0$$
$$\lim_{t \to \infty} \tau^*(t; X, Y) = 0.$$

- As a result, it follows that the maximum of the function $\tau^*(t; X, Y)$ will not be attained for very small or very large values of the threshold t.
- Therefore, $\Theta_\infty^*(X, Y)$ is not sensitive to the extreme events in the tail because the threshold t, for which $P(X \leq t < Y)$ is maximal, is near the center of the distributions.

Limit cases of $\mathcal{L}_p^*(X, Y)$ and $\Theta_p^*(X, Y)$

$\Rightarrow$ Exactly the same effect is present in the minimal quasi-semi-metric generated by it, $\theta_p^*(X, Y)$.

- As p increases to infinity, we obtain

$$\theta_\infty^*(X, Y) = \sup_{t \in R}[\max(F_X(t) - F_Y(t), 0)]$$

  which is an asymmetric version of the celebrated Kolmogorov metric.

- Basically, $\theta_\infty^*(X, Y)$ calculates the maximal difference between the distribution functions, $F_X(t) - F_Y(t)$.
- Therefore, $\theta_\infty^*(X, Y)$ is not sensitive to the deviations between the two distribution functions in the tails, which describe the probability of extreme events, because as t approaches either of the infinities, the difference $F_X(t) - F_Y(t)$ decays to zero.

Computing r.d. metrics in practice

- Here we'll state a number of closed form expressions for some of the r.d. metrics considered in the previous sections and we give examples in the setting of the benchmark-tracking problem.
- Generally, it is not possible to arrive at a closed form expression but under additional assumptions for the joint distribution of the pair of random variables, explicit representations can be provided.

- Suppose that (X, Y) has a centered, bivariate normal distribution. In this case, the difference $Y - X$ has a zero-mean, normal distribution with standard deviation $\sigma(Y - X)$, $Y - X \in N(0, \sigma^2(Y - X))$.
- The difference has the same distribution as $\sigma(Y - X)Z$, where $Z \in N(0, 1)$. We use this representation only to calculate the expectation. In effect, we obtain

$$\mathcal{L}_p^*(X, Y) = C_p.\sigma(Y - X), \quad p \geq 1 \tag{5}$$

where $C_p = (E(\max(Z, 0))^p)^{1/p}$ is a positive constant which can be explicitly computed.

- Note that the parameter p influences the constant C_p only and, therefore, $\mathcal{L}_p^*(X, Y)$ is just a scaled standard deviation of the difference $Y - X$.
- This is not true only under the hypothesis of joint normal distribution. If (X, Y) has a joint elliptical distribution with finite variance, then $\mathcal{L}_p^*(X, Y)$ has, essentially, the form given by (5).
- In the elliptical case, one has to ensure additionally that X and Y have finite p-th absolute moment, i.e. $E|X|^p < \infty$ and $E|Y|^p < \infty$. Otherwise, $\mathcal{L}_p^*(X, Y)$ may become infinite.

Explicit form of $\mathcal{L}_p^*(X, Y)$ when (X, Y) has joint normal distribution

- Apparently, the closed-form expression (5) can be regarded as typical of the large class of bivariate elliptical distributions in which the joint normal distribution is just a special case.
- It may seem strange that even though by definition the r.d. metric $\mathcal{L}_p^*(X, Y)$ is asymmetric, equation (5) is symmetric. The reason is the elliptical assumption because it implies symmetric distributions of X, Y, and the difference $Y - X$ and, therefore, $\mathcal{L}_p^*(X, Y)$ cannot be asymmetric because of this restrictive assumption.

Let us apply equation (5) to the benchmark-tracking problem.

- To this end, we interpret the r.v. X as the portfolio return r_p and the random variable Y as the benchmark return r_b.
- Concerning the random vector of assets returns, we assume that it follows the multivariate normal, or multivariate elliptical, in order to make sure that the distribution of the portfolio return r_p is normal, or elliptical, for any choice of portfolio weights.
- As a result, the r.d. metric has the form,

$$\mathcal{L}_p^*(r_{p0}, r_{b0}) = C_p.\sigma(r_p - r_b), \quad p \geq 1$$

  which means that $\mathcal{L}_p^*(r_{p0}, r_{b0})$ is just a scaled tracking error.
- The tracking error is the building block of the $\mathcal{L}_p^*(r_{p0}, r_{b0})$ r.d. metric in the multivariate normal, or, more generally, in the multivariate elliptical case. This will not happen under the assumption of a multivariate skewed distribution.

Explicit form of $\ell_p^*(X,Y)$ when X and Y have normal distribution

- The minimal r.d. metric $l_p^*(X,Y)$ is simple and, therefore, we do not need a distributional assumption for the pair (X,Y) but only for the marginal laws of X and Y.
- Suppose that both X and Y have the centered normal distribution, $X \in N(0,\sigma_X^2)$ and $Y \in N(0,\sigma_Y^2)$. Both distributions can be represented as a scaled $N(0,1)$ distributions and, as a result, we obtain

$$\ell_p^*(X,Y) = C_p.|\sigma_X - \sigma_Y|, \quad p \geq 1 \tag{6}$$

where $C_p = (E(\max(Z,0))^p)^{1/p}$ and $Z \in N(0,1)$.

Explicit form of $\ell_p^*(X, Y)$ when X and Y have normal distribution

- In the setting of the benchmark-tracking problem, assume additionally that r follows the multiavriate normal distribution with covariance matrix Σ, $r_0 \in N(0, \Sigma)$.
- We obtain the explicit formula

$$\ell_p^*(r_0^b, w'r_0) = C_p.|\sqrt{w'\Sigma w} - \sigma(r_b)|, \quad p \geq 1.$$

  where w denotes the vector of portfolio weights.
- In the lecture, we provided the case $p = 1$ in which $C_1 = 1/\sqrt{2\pi}$.

Explicit forms of $\boldsymbol{\Theta}_p^*(X,Y)$ and $\theta_p^*(X,Y)$

- It is much harder to calculate closed-form expressions for $\boldsymbol{\Theta}_p^*(X,Y)$ and $\boldsymbol{\theta}_p^*(X,Y)$ even under additional assumptions for the joint distribution of (X,Y).
- Nevertheless, for some choices of p, it is possible to link the two r.d. metrics to other classes for which the calculation is not so complicated.
- For instance, $\boldsymbol{\Theta}_1^*(X,Y) = \mathcal{L}_1^*(X,Y)$ and $\boldsymbol{\theta}_1^*(X,Y) = \ell_1^*(X,Y)$ and we can use the already derived explicit forms.

Estimating r.d. metrics from a sample

*How can we calculate the simple r.d. metrics **(7)** and **(8)** in the lecture for a given portfolio with weights $w = (w_1, \ldots, w_n)$ if we have a sample of daily observations for the equity returns and the benchmark returns?*

- Notice that they both involve either the distribution functions of the portfolios returns or the inverse of the distribution functions and these functions we do not know in practice.
- They have to be estimated either directly from the data making no distributional hypotheses, or assuming a parametric model and estimating its parameters from the sample.

Estimating r.d. metrics from a sample

We may assume that the equity returns and the benchmark returns are jointly distributed according to the multivariate normal distribution.

- It follows that the equity returns also have the multivariate normal distribution and, consequently, the return r_p of any portfolio with weights w also has the normal distribution, $r_p \in N(0, w'\Sigma w)$, where Σ is the covariance matrix of the equity returns.
- As a result, in order to calculate **(7)** and **(8)**, we have to estimate the unknown parameters in the first place; that is, the covariance matrix Σ and the variance of the benchmark returns, $\sigma^2(r_b)$.
- Once we know the estimates $\hat{\Sigma}$ and $\hat{\sigma}^2(r_b)$, we can calculate **(7)** by plugging in the distribution functions of the centered normal distribution with variance equal to the corresponding estimates.
- The integrals can be calculated numerically using an available software package such as MATLAB.

Estimating r.d. metrics from a sample

- The r.d. metric **(8)** can be calculated by by taking advantage of the closed-form expression (6),

$$\hat{\ell}_p^*(r_{p0}, r_{b0}) = C_p \left| (w'\hat{\Sigma}w)^{1/2} - \sigma(\hat{r}_b) \right|$$

- Note that when $p = 1$, we have the following special case,

$$\hat{\boldsymbol{\theta}}_1^*(r_{p0}, r_{b0}) = \hat{\ell}_1^*(r_{p0}, r_{b0}) = \frac{1}{\sqrt{2\pi}} \left| (w'\hat{\Sigma}w)^{1/2} - \sigma(\hat{r}_b) \right|.$$

Estimating r.d. metrics from a sample

- Suppose that we do not want to make any distributional hypotheses. Then, the r.d. metrics can be computed through the empirical distribution functions and the empirical inverse distribution functions.
- Thus, in the case of **(12)** from the lecture, we use

$$\hat{\theta}_p^*(r_{p0}, r_{b0}) = \left[\int_{-\infty}^{\infty} (\max(\widehat{F}_{r_{b0}}(t) - \widehat{F}_{r_{p0}}(t), 0))^p dt\right]^{1/p}, \; p \geq 1$$

  where $\widehat{F}_X(t) = \frac{1}{n}\sum_{i=1}^{n} I_{\{X_i \leq t\}}$, denotes the empirical distribution function and n is the sample size.
- The integral can be calculated numerically using an available software package.

Estimating r.d. metrics from a sample

- The empirical r.d. metric **(13)** can be easily simplified because the stocks in our sample and the index have the same number of observations.
- In order to give the formula, we introduce additional notation. Let us fix the portfolio weights w, then denote by $r_{p0}^{(1)} \leq r_{p0}^{(2)} \leq \ldots \leq r_{p0}^{(n)}$ the sorted sample of the corresponding observed centered portfolio returns. Similarly, let $r_{b0}^{(1)} \leq r_{b0}^{(2)} \leq \ldots \leq r_{b0}^{(n)}$ be the sorted sample of the observed centered benchmark returns. Then

$$\widehat{\ell}_p^*(r_{p0}, r_{b0}) = \left[\frac{1}{n} \sum_{i=1}^{n} (\max(r_{b0}^{(i)} - r_{p0}^{(i)}, 0))^p \right]^{1/p}, \; p \geq 1.$$

- From the point of view of computational burden, minimizing $\widehat{\ell}_p^*(r_{p0}, r_{b0})$ is a lot easier than $\widehat{\theta}_p^*(r_{p0}, r_{b0})$ because the numerical integration adds more complexity to the problem.

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