Technical Appendix

Lecture 8: Optimal portfolios

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The material is based on the text-book:

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Types of constraints

The kinds of constraints defining the set of feasible portfolios do not depend on whether we consider M-V analysis, M-R analysis, or a more general framework, and are determined by exogenous factors.

- A risk-averse portfolio manager would not want to see a high concentration of a particular, or any, asset in the portfolio.
- For some of the assets a minimal holding may be required.
- These two conditions can be implemented by means of box-type constraints,

$$a_i \leq w_i \leq b_i, \qquad i = 1, 2, \ldots, n,$$

  where a_i is a lower bound and b_i is an upper bound on the weight of the i-th asset.

Types of constraints

For some assets, the lower bound can simply be zero, $a_i = 0$.

- For example, suppose there are 3 assets in the investment universe and we want to invest no more than 60% of the capital in any of them and in asset number 3 to invest at least 10%.
- This is modeled by the constraints,

$$\begin{aligned} 0 \leq & w_1 \leq 0.6 \\ 0 \leq & w_2 \leq 0.6 \\ 0.1 \leq & w_3 \leq 0.6. \end{aligned}$$

Types of constraints

In defining box-type constraints, we have to be careful not to end up with an overly stringent set of constraints.

- For example, this happens in the above illustration if the upper bound is 20% instead of 60%.
- Since all weights have to sum up to one, the sum of lower bounds should not be above 1, $\sum_{i=1}^{n} a_i \leq 1$, and the sum of upper bound should not be below 1, $\sum_{i=1}^{n} b_i \geq 1$.

Types of constraints

In a similar manner, the portfolio manager may want to impose constraints on certain groupings of assets.

- Suppose that the investment universe consists of common stocks. Depending on the strategy type, a reasonable condition is a lower and an upper bound on the exposure in a given industry.
- This is a constraint on the sum of the weights corresponding to all stocks from the investment universe belonging to that industry,

$$a \leq \sum_{i \in I} w_i \leq b,$$

in which I denotes the indices of the common stocks from the given industry.

Types of constraints

- The general rule which is followed when building constraints is that the resulting set of feasible portfolios should be *convex*.
- This is guaranteed if each of the inequalities or equalities building up the constraints define a convex set. Then, the set of feasible portfolios is the intersection of these convex sets, which in turn is a convex set.

Types of constraints

- If the set of feasible portfolios is *not convex*, then the optimization problem may become hard to solve numerically.
- An example of a type of constraint which does not lead to a convex set of feasible portfolios is the following.
- In the example above, suppose that if an asset is to be included in the portfolio, then it should have at least 10% of the capital allocated to it. This is modeled by the constraints

$$\begin{aligned} w_1 = 0 \quad &\text{or} \quad 0.1 \leq w_1 \leq 0.6 \\ w_2 = 0 \quad &\text{or} \quad 0.1 \leq w_2 \leq 0.6 \\ w_3 = 0 \quad &\text{or} \quad 0.1 \leq w_3 \leq 0.6 \end{aligned}$$

  which do not result in a convex set.

$\Rightarrow$ Problems of this type can be solved by the more general methods of mixed-integer programming and can be very computationally intensive.

Types of constraints

- The set of feasible portfolios has a simpler structure if it contains only linear inequalities or equalities.
- In this case, it is said to be polyhedral. Every polyhedral set is convex since any linear inequality or equality defines a convex set.
- A polyhedral set has a simpler structure because its borders are described by hyperplanes, which is a consequence of the fact that the set is composed of linear inequalities or equalities.

Quadratic approximations to utility functions

- We remarked that M-V analysis is, generally, inconsistent with SSD.
- It is consistent with the order implied by investors with quadratic utility functions. The assumption that investors have quadratic utility functions is a significant limitation.
- Under certain conditions, quadratic utility functions may represent a reasonable approximation of more general types of utility functions.
- Therefore, there are cases in which the decisions made by investors with quadratic utilities are consistent with the decisions made by larger classes of investors depending on the accuracy of the approximation.

Quadratic approximations to utility functions

- Consider a utility function $u(x)$ and its Taylor series approximation in a neighborhood of the point EX where X is a random variable,

$$u(x) = u(EX) + u'(EX)(x - EX) + \frac{1}{2}u''(EX)(x - EX)^2 + \frac{1}{n!}\sum_{k=3}^{\infty} u^{(k)}(EX)(x - EX)^k, \tag{1}$$

  where $u^{(k)}(x)$ denotes the k-th derivative of $u(x)$ and x is in a neighborhood of the point EX.
- We assume that the infinite series expansion is valid for any $x \in \mathbb{R}$; that is, the infinite power series converges to the value $u(x)$ for any real x.
- This condition is already a limitation on the possible utility functions that we consider.
- Not only do we require that the utility function is infinitely many times differentiable but we also assume that the corresponding Taylor expansion is convergent for any real x. Functions satisfying these conditions are called analytic functions.

Quadratic approximations to utility functions

- We can calculate the expected utility taking advantage of the expansion in (1) which we integrate term by term,

$$Eu(X) = u(EX) + \frac{1}{2}u''(EX)E(X - EX)^2 + \frac{1}{n!}\sum_{k=3}^{\infty} u^{(k)}(EX)E(X - EX)^k. \tag{2}$$

- The second term vanishes because $E(X - EX) = 0$.
- We obtain that the expected utility can be expressed in terms of the derivatives of the utility function evaluated at EX and all moments $m_k = E(X - EX)^k$, $k = 1, 2, \ldots$
- Even for analytic utilities $u(x)$, expression (2) may not hold.
- If the r.v. X has infinite moments, then (2) does not hold. Therefore, a critical assumption is that the r.v. X has finite moments of any order.

Quadratic approximations to utility functions

- If $u(x)$ is analytic and the r.v. X is such that $m_k < \infty$, $k = 1, 2, \ldots$, then we may choose the first three terms as a reasonable approximation,

$$\begin{aligned} Eu(X) &\approx u(EX) + \frac{1}{2}u''(EX)E(X - EX)^2 \\ &= u(EX) + \frac{1}{2}u''(EX)\sigma_X^2, \end{aligned} \tag{3}$$

  for the expected utility function.
- We recognize the moment $\sigma_X^2 = E(X - EX)^2$ as the variance of X.
- The expected utility is approximated by the mean and the variance of X.
- If we consider risk-averse investors, then the utility function $u(x)$ is concave and, therefore, it has a negative second derivative. As a result, the expected utility maximization problem can be linked to M-V analysis.

Quadratic approximations to utility functions

- Samuelson (1970) shows that under certain conditions, the approximation in (3) is reasonable.
- If the choice under uncertainty concerns a very short interval of time and the random variable describes the payoff of a venture at the end of the time period, then under a few regularity conditions the approximation in (3) holds.
- Ohlson (1975) considers weaker conditions under which (3) is reasonable.

Solving mean-variance problems in practice

The main optimization problems behind M-V analysis are **(4)**, **(5)**, and **(8)**, given in the lecture.

- In problem **(4)**, the portfolio variance is minimized with a constraint on the expected return.
- The objective function of this problem is quadratic and if the set of feasible portfolios is polyhedral, then the optimization problem is said to be a quadratic programming problem.

Solving mean-variance problems in practice

- Problem **(5)** has a more simple objective as we maximize the expected portfolio return which is a linear function of portfolio weights.
- In the set of feasible portfolios, we include an upper bound on the portfolio variance which results in a quadratic constraint.
- If all the other constraints are linear, the optimization problems can be formulated as second order cone programming problems.

Solving mean-variance problems in practice

- Finally, problem **(8)** of the lecture is very similar in structure to **(4)**.
- The objective function is also quadratic, the difference from **(4)** is that it has a linear part represented by the expected portfolio return.
- **(8)** is a quadratic programming problem.
- As far as the computational complexity is concerned, the quadratic and, more generally, the conic programming problems are between the linear optimization problems and the convex optimization problems with non-linear constraints.

Solving mean-variance problems in practice

Under certain conditions with no inequality constraints, it is possible to obtain a closed-form solution to mean-variance optimization problems.

- For example, the optimization problem

$$\begin{aligned} \min_{w} \quad & w'\Sigma w \\ \text{subject to} \quad & w'e = 1 \\ & w'\mu = R_*, \end{aligned} \tag{4}$$

  which is a simplified analogue of **(4)** given in the lecture, allows for a closed-form solution. We have replaced the inequality constraint on the expected portfolio return by an equality constraint and we have removed the requirement that the weights should be non-negative.

- As a result, closed-form solution to mean-variance optimization problems allows for taking a short position in an asset, which is indicated by a negative weight in the optimal solution.

Solving mean-variance problems in practice

The closed-form solution is obtained by applying the method of Lagrange multipliers which is as follows.

- *First*, we build the corresponding Lagrangian represented by the function

$$L(w,\lambda) = w'\Sigma w + \lambda_1(1 - w'e) + \lambda_2(R_* - w'\mu)$$

  in which λ_1 and λ_2 are the Lagrange multipliers.

- *Second*, we solve for w the system of equations resulting from the first-order optimality conditions of the Lagrangian,

$$\left| \begin{array}{l} \dfrac{\partial L(w,\lambda)}{\partial w} = 0 \\[2ex] \dfrac{\partial L(w,\lambda)}{\partial \lambda} = 0. \end{array} \right.$$

Solving mean-variance problems in practice

- Since the Lagrangian is a quadratic function of w, the resulting system of equations is composed of linear equations which can be solved for w.
- Then we obtain that the optimal solution can be computed according to the formula in matrix form

$$w = \frac{(C\Sigma^{-1}\mu - B\Sigma^{-1}e)m + A\Sigma^{-1}e - B\Sigma^{-1}\mu}{AC - B^2}$$

where Σ^{-1} stands for the inverse of the covariance matrix Σ, $A = \mu'\Sigma^{-1}\mu$, $B = e'\Sigma^{-1}\mu$, and $C = e'\Sigma^{-1}e$.

Solving mean-variance problems in practice

- If there are inequality constraints, this Lagrange multipliers approach is not applicable.
- In this case, the optimization problem is more general and the Karush-Kuhn-Tucker conditions, which generalize the method of Lagrange multipliers, can be applied but they rarely lead to nice closed-form expressions as the resulting system of equations is much more involved.

Solving mean-risk problems in practice

- The optimization problems arising from M-R analysis have a different structure than the quadratic problems of M-V analysis which depends on the assumed properties of the risk measure.
- In Lecture 5, we considered two classes of risk measures introduced axiomatically.
- Generally, the most important property which determines to a large extent the structure of the optimization problem is the convexity property. It guarantees the diversification effect; that is, the risk of a portfolio of assets is smaller than the corresponding weighted average of the individual risks.

Solving mean-risk problems in practice

- Under the general assumption of convexity, problems **(12)**, **(13)**, and **(18)** in the lecture are convex programming problems.
- In **(12)** and **(18)**, the objective functions are convex functions and in **(13)**, there is a convex function in the constraint set.
- The three problems can be solved in practice using the general methods of convex programming.
- There are commercial solvers (MATLAB, for example) in which such algorithms are implemented.

Solving mean-risk problems in practice

- Under certain conditions, simplification of the optimization problem structure is possible for some risk measures.
- If we choose AVaR as a risk measure, then the three problems can be reduced to linear optimization problems provided that future scenarios are available.
- In the lecture, we demonstrated that **(16)** and **(17)** correspond to **(12)** and **(13)** and both **(16)** and **(17)** have linear objective functions and the set of feasible portfolios is defined through linear inequalities and equalities.
- Both problems are linear programming problems which are significantly simpler than a convex optimization problem.

Solving mean-risk problems in practice

However, reducing the convex problem to a linear problem comes at the cost of increasing the problem dimension.

- For instance, problem **(12)** in the lecture has n variables and $n+2$ linear constraints, where n denotes the number of assets in the portfolio.
- In contrast, the corresponding linear problem **(16)** in the lecture has $n+k+1$ variables and $2k+n+2$ linear constraints, in which k denotes the number of scenarios.

$\Rightarrow$ The dimension of the linear problem increases with the number of scenarios because we introduce one auxiliary variable and two constraints for each scenario.

Solving mean-risk problems in practice

- Adding more scenarios makes the matrix defining the linear constraints in the linear programming problem become more non-sparse.
- A matrix is called sparse if most of the numbers in it are zeros and the numerical methods for linear programming are more efficient if the matrix is more sparse.

$\Rightarrow$ We are simplifying the problem structure but we are increasing the problem dimension.

Solving mean-risk problems in practice

- When increasing the number of scenarios there will be a point at which the two effects will balance off and there will not be an advantage in solving the linear problem.
- In this case, one may consider directly

$$\begin{array}{rl} \min\limits_{w} & \widehat{AVaR}_{\epsilon}(Hw) \\ \text{subject to} & w'e = 1 \\ & w'\mu \geq R_{*} \\ & w \geq 0. \end{array} \tag{5}$$

  in which $\widehat{AVaR}_{\epsilon}(Hw)$ is the sample AVaR and H is the matrix with scenarios defined in **(14)** in the lecture.
- Problem (5) can be solved as a convex programming problem.

Solving mean-risk problems in practice

There is another way of viewing problems (5) and **(12)** (**(12)** is given in the lecture) with $\rho(r_p) = AVaR_\epsilon(r_p)$.

- Suppose that we know exactly the multivariate distribution of the assets returns but we cannot obtain explicitly the AVaR risk measure as a function of portfolio weights.
- However, we have a random number generator constructed that we can use to draw independent scenarios from the multivariate law. Then we cannot solve **(12)** because we cannot evaluate the objective function for a given vector of portfolio weights.
- Nevertheless, we can draw a matrix of independent simulations from the multivariate law and compute *approximately* the AVaR for any vector of portfolio weights through the formula of the sample AVaR.
- Thus, we can solve problem (5), or **(16)**, which can be viewed as an approximation to **(12)** obtained through the Monte Carlo method. The larger the number of scenarios, the more accurate the approximation.

Solving mean-risk problems in practice

- Also, the larger the portfolio, the more simulations we need to achieve a given level of accuracy since the generated vectors are supposed to approximate a distribution in a higher dimensional space.
- Therefore, the linear problem **(16)** in the lecture may not be advantageous if higher accuracy is needed or, alternatively, if the portfolio is sufficiently large and there is a target accuracy.
- One can use (5) in which directly the sample AVaR is getting minimized without increasing the problem dimension by including additional variables and constraints.

Reward-risk analysis

- In M-R analysis we consider two criteria as a major determinant of efficient portfolios:
  - expected portfolio return, being a measure of the expected performance,
  - a risk measure estimating portfolio risk.
- Instead of the expected portfolio return, we can include a more general functional $\nu(X)$ estimating expected performance.
- We can generalize M-R analysis by considering $\nu(X)$ and the risk measure $\rho(X)$ as criteria for obtaining efficient portfolios.
- M-R analysis appears as a special case when $\nu(X) = EX$. The functional $\nu(X)$ we call a reward measure and the resulting more general analysis is called reward-risk analysis (R-R analysis).

Reward-risk analysis

We impose several properties on the functional $\nu(X)$ and explore the resulting optimization problems. Consider the following properties.

1. Monotonicity. Suppose that $X \leq Y$ in almost sure sense. It is reasonable to expect that the expected reward of Y will be larger than that of X, $\nu(X) \leq \nu(Y)$.
2. Superadditivity. We assume that for any X and Y, the following inequality holds,

$$\nu(X + Y) \geq \nu(X) + \nu(Y).$$

   That is, the reward of a portfolio is not smaller than the sum of the portfolio constituents rewards. There is an additional stimulus in holding a portfolio.

3. Positive homogeneity. The rationale of this assumption is the same as in the case of risk measures.

$$\nu(hX) = h\nu(X), \qquad h \geq 0.$$

4. Invariance property. Adding a non-random term to the portfolio increases the reward by the non-random quantity,

$$\nu(X + C) = \nu(X) + C,$$

and $\nu(0) = 0$.

Reward-risk analysis

- These axioms suggest that the negative of a coherent risk measure is in fact a reward measure; that is, if $\nu(X) = -\rho(X)$ where $\rho(X)$ is a coherent risk measure, then the above properties hold.
- If $\nu(X)$ satisfies the properties above, we call it a coherent reward measure.
- The superadditivity and the positive homogeneity properties guarantee that any coherent reward measure is a concave function,

$$\nu(aX + (1-a)Y) \geq a\nu(X) + (1-a)\nu(Y),$$

  where $a \in [0, 1]$.
- This property, along with the convexity of the risk measure, guarantees nice properties of the resulting optimization problems.

Reward-risk analysis

The main principles of R-R analysis can be formulated in the same way as for M-R analysis.

1. From all feasible portfolios with a given lower bound on the reward measure, find the portfolios that have minimum risk.
2. From all feasible portfolios with a given upper bound on risk, find the portfolios that provide maximum reward.

Reward-risk analysis

- The corresponding optimal portfolio problems are the following

$$\begin{array}{rl} \min\limits_{w} & \rho(r_p) \\ \text{subject to} & w'e = 1 \\ & \nu(r_p) \geq R_* \\ & w \geq 0, \end{array} \tag{6}$$

  where R_* is the lower bound on the portfolio reward.

- The the optimization problem behind the second formulation is

$$\begin{array}{rl} \max\limits_{w} & \nu(r_p) \\ \text{subject to} & w'e = 1 \\ & \rho(r_p) \leq R^* \\ & w \geq 0, \end{array} \tag{7}$$

  where R^* is the upper bound on portfolio risk.

- Problem (6) is a convex optimization problem and (7) is reducible to a convex problem by flipping the sign of the objective function and considering minimization.
- Convex optimization problems are appealing because a local minimum is necessarily the global one. The necessary and sufficient conditions are given by the Karush-Kuhn-Tucker theorem.

Reward-risk analysis

- The optimal solutions obtained from the two problems by varying the limits on the portfolio reward or risk respectively are called reward-risk efficient portfolios.
- The coordinates of the reward-risk efficient portfolios in the reward-risk plane form the reward-risk efficient frontier. It is a concave, monotonically increasing function if the reward measure is a concave function, and the risk measure is a convex function.
- The general shape of the reward-risk efficient frontier is the same as the one plotted in *Figures 4,5* in the lecture.
- As a consequence of the Karush-Kuhn-Tucker conditions, the efficient frontier can also be generated by the problem

$$\begin{array}{rl} \max\limits_{w} & \nu(r_p) - \lambda\rho(r_p) \\ \text{subject to} & w'e = 1 \\ & w \geq 0, \end{array} \tag{8}$$

where $\lambda \geq 0$ is the risk-aversion parameter, or the Lagrange multiplier.

We will demonstrate that the reward-risk efficient portfolios can be derived from a reward-dispersion optimal portfolio problem.

- Consider the optimization problem (8).
- The objective function is transformed in the following way,

$$\begin{aligned}\nu(r_p) - \lambda\rho(r_p) &= \nu(r_p) - \lambda\rho(r_p - \nu(r_p) + \nu(r_p)) \\ &= (\lambda + 1)\nu(r_p) - \lambda\rho(r_p - \nu(r_p)) \\ &= (\lambda + 1)\left(\nu(r_p) - \frac{\lambda}{(\lambda + 1)}\rho(r_p - \nu(r_p))\right).\end{aligned}$$

- The positive multiplier $\lambda + 1$ does not change the optimal solutions and we can safely ignore it.

- As a result, we obtain the equivalent optimization problem

$$\begin{array}{rl} \max\limits_{w} & \nu(r_p) - \dfrac{\lambda}{(\lambda+1)}\rho(r_p - \nu(r_p)) \\ \text{subject to} & w'e = 1 \\ & w \geq 0, \end{array} \quad (9)$$

  where $\lambda \geq 0$ and, as a result, the multiplier $\lambda/(\lambda+1) \in [0,1)$.
- The functional $G(X) = \rho(X - \nu(X))$ is a dispersion measure under the additional condition $\rho(X) \geq -\nu(X)$, as it satisfies the general properties outlined, which we illustrate below.

- Positive shift
  $G(X+C) = \rho(X + C - \nu(X+C)) = G(X)$ for all X and constants $C \in \mathbb{R}$
- Positive homogeneity
  $G(0) = \rho(0 - \nu(0)) = 0$ and $G(hX) = \rho(hX - \nu(hX)) = hG(X)$ for all X and all $h > 0$
- Positivity
  Under the additional condition $\rho(X) \geq -\nu(X)$, it follows directly that $G(X)$ is positive, $G(X) \geq 0$ for all X, with $G(X) > 0$ for non-constant X, from the representation

$$G(X) = \rho(X - \nu(X)) = \rho(X) + \nu(X).$$

- As a result, we can consider the more general reward-dispersion optimal portfolio problem

$$\begin{array}{rl} \max\limits_{w} & \nu(r_p) - aG(r_p) \\ \text{subject to} & w'e = 1 \\ & w \geq 0, \end{array} \tag{10}$$

where $a \geq 0$ and $G(X) = \rho(X - \nu(X))$.

- The reward-risk efficient portfolios are obtained from (10) with $a \in [0, 1]$.
- The optimal portfolios obtained with $a > 1$ are in addition to the mean-risk efficient portfolios and are sub-optimal according to R-R analysis.

Reward-risk analysis

- Note that problem (10) may not be a convex optimization problem for all values of a because the functional G is, generally, arbitrary as it equals a sum of a convex and a concave functional.
- However, if $a \in [0, 1]$ then (10) is a convex optimization problem because it is equivalent to (9).
- As the dispersion measures can be derived from probability metrics, the set of efficient portfolios can be related to the theory of probability metrics through the reward-dispersion optimization problem.

Reward-risk analysis

- In the lecture, we showed a special case of this relationship in which the reward measure equals the expected portfolio return and the $\rho(X)$ is a coherent risk measure satisfying $\rho(X) \geq -EX$.
- Under these conditions, the functional $G(X)$ turns into a deviation measure, which is an example of a dispersion measure, and the corresponding problem (10) has better optimal properties.

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Chapter 8.