Lecture 8: Optimal portfolios

Prof. Dr. Svetlozar Rachev

Institute for Statistics and Mathematical Economics
University of Karlsruhe

Portfolio and Asset Liability Management

Summer Semester 2008

Copyright

The material is based on the text-book:

Svetlozar T. Rachev, Stoyan Stoyanov, and Frank J. Fabozzi
Advanced Stochastic Models, Risk Assessment, and Portfolio Optimization: The Ideal Risk, Uncertainty, and Performance Measures
John Wiley, Finance, 2007

Prof. Svetlozar (Zari) T. Rachev
Chair of Econometrics, Statistics
and Mathematical Finance
School of Economics and Business Engineering
University of Karlsruhe
Kollegium am Schloss, Bau II, 20.12, R210
Postfach 6980, D-76128, Karlsruhe, Germany
Tel. +49-721-608-7535, +49-721-608-2042(s)
Fax: +49-721-608-3811
http://www.statistik.uni-karslruhe.de

- A portfolio is a collection of investments held by an institution or a private individual.
- Portfolios are constructed and held as a part of an investment strategy and for the purpose of diversification. The concept of diversification: Including a number of assets in a portfolio may greatly reduce portfolio risk while not necessarily reducing performance.
- The problem of choosing a portfolio is a problem of choice under uncertainty because the payoffs of financial instruments are uncertain.
- An optimal portfolio is a portfolio which is most preferred in a given set of feasible portfolios by an investor or a certain category of investors.

Introduction

- Investors' preferences are characterized by utility functions and they choose the venture yielding maximum expected utility.
- As a consequence of the theory, stochastic dominance relations arise, describing the choice of groups of investors, such as the risk-averse investors.
- While the foundations of expected utility theory as a normative theory are solid, its practical application is limited as the resulting optimization problems are very difficult to solve.
- For example, given a set of feasible portfolios, it is hard to find the ones which will be preferred by all risk-averse investors by applying directly the characterization in terms of the cumulative distribution functions (c.d.f.s).

- A different approach towards the problem of optimal portfolio choice was introduced by Harry Markowitz in the 1950, mean-variance analysis (M-V analysis) and popularly referred to as modern portfolio theory.
- He suggested that the portfolio choice be made with respect to two criteria: the expected portfolio return and the variance of the portfolio return, the latter used as a proxy for risk.
- A portfolio is preferred to other portfolio one if it has higher expected return and lower variance.
- M-V analysis is easy to apply in practice. There are convenient computational recipes for the resulting optimization problems and geometric interpretations of the trade-off between the expected return and variance.

Introduction

If all risk-verse investors identify a given portfolio as most preferred, then is the same portfolio identified by M-V analysis also optimal?

- Basically, the answer to this question is negative.
- M-V analysis is not consistent with second-order stochastic dominance (SSD) unless the joint distribution of investment returns is multivariate normal, which is a very restrictive assumption.
- Alternatively, M-V analysis describes correctly the choices made by investors with quadratic utility functions.
- Again, the assumption of quadratic utility functions is very restrictive even though we can extend it and consider all utility functions which can be sufficiently well approximated by quadratic utilities.

- Another well-known drawback is that in M-V analysis, variance is used as a proxy for risk. In Lecture 5, we demonstrated that variance is not a risk measure but a measure of uncertainty.
- This deficiency was recognized by Markowitz (1959) and he suggested the downside semi-standard deviation as a proxy for risk. In contrast to variance, the downside semi-standard deviation is consistent with SSD.
- If the risk measure is consistent with SSD, so is the optimal solution to the optimization problem.
- the optimization problem is appealing from a practical viewpoint because it is computationally feasible and there are similar geometric interpretations as in M-V analysis. We call this generalization mean-risk analysis (M-R analysis).

- The classical mean-variance framework is the first proposed model of the reward-risk type. The expected portfolio return is used as a measure of reward and the variance of portfolio return indicates how well-diversified the portfolio is. Lower variance means higher diversification level.

$\Rightarrow$ The portfolio choice problem is typically treated as a one-period problem.

- Suppose that at time $t_0 = 0$ we have an investor who can choose to invest among a universe of n assets.
- Having made the decision, he keeps the allocation unchanged until the moment t_1 when he can make another investment decision based on the new information accumulated up to t_1. In this sense, it is also said that the problem is static, as opposed to a dynamic problem in which investment decisions are made for several time periods ahead.

The main principle behind M-V analysis can be summarized in two ways:

1. From all feasible portfolios with a given lower bound on the expected performance, find the ones that have the minimum variance (i.e., the maximally diversified ones).
2. From all feasible portfolios with a given upper bound on the variance of portfolio return (i.e., with an upper bound on the diversification level), find the ones that have maximum expected performance.

Mean-variance analysis

- There could be certain limitations for the feasible portfolio, these limitations can be strategy specific.
- For example, there may be constraints on the maximum capital allocation to a given industry, or a constraint on the correlation with a given market segment.
- The limitations can also be dictated by liquidity considerations, for instance a maximum allocation to a given position, constraints on transaction cost or turnover.

Mean-variance optimization problems

We can find two optimization problems behind the formulations of the main principle of M-V analysis.

- We will use matrix notation to make the problem formulations concise.
- Suppose that the investment universe consists of n financial assets. Denote the assets returns by the vector $X' = (X_1, \dots, X_n)$ in which X_i stands for the return on the i-th asset.
- The returns are random and their mean is denoted by $\mu' = (\mu_1, \dots, \mu_n)$ where $\mu_i = EX_i$. The returns are also dependent on each other in a certain way.
- The dependence will be described by the covariances. Between the i-th and the j-th return it is denoted by

$$\sigma_{ij} = \mathrm{cov}(X_i, X_j) = E(X_i - \mu_i)(X_j - \mu_j).$$

- σ_{ii} stands for the variance of the return of the i-th asset,

$$\sigma_{ii} = E(X_i - \mu_i)^2.$$

Mean-variance optimization problems

- The result of an investment decision is a portfolio, the composition of which is denoted by $w' = (w_1, \ldots, w_n)$, where w_i is the portfolio weight corresponding to the i-th instrument.
- We will consider long-only strategies which means that all weights should be non-negative, $w_i \geq 0$, and should sum up to one,

$$w_1 + w_2 + \ldots + w_n = w'e = 1.$$

where $e' = (1, 1, \ldots, 1)$. These conditions will be set as constraints in the optimization problem.

Mean-variance optimization problems

- The return of a portfolio r_p can be expressed by means of the weights and the returns of the assets,

$$r_p = w_1 X_1 + w_2 X_2 + \ldots + w_n X_n = \sum_{i=1}^{n} w_i X_i = w' X . \tag{1}$$

- Similarly, the expected portfolio return can be expressed by the vector of weights and expected assets returns,

$$E r_p = w_1 \mu_1 + w_2 \mu_2 + \ldots + w_n \mu_n = \sum_{i=1}^{n} w_i \mu_i = w' \mu . \tag{2}$$

- Finally, the variance of portfolio returns σ_p^2 can be expressed by means of portfolio weights and the covariances σ_{ij} between the assets returns,

$$\begin{aligned} \sigma_{r_p}^2 &= E(r_p - E r_p)^2 \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} . \end{aligned}$$

- The covariances of all asset returns can be arranged in a matrix and $\sigma_{r_p}^2$ can be expressed as

$$\sigma_{r_p}^2 = w'\Sigma w \tag{3}$$

where Σ is a $n \times n$ matrix of covariances,

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{pmatrix}.$$

Mean-variance optimization problems

- The optimization problem behind the *first formulation* of the main principle of M-V analysis is

$$\begin{array}{rl} \min\limits_{w} & w'\Sigma w \\ \text{subject to} & w'e = 1 \\ & w'\mu \geq R_* \\ & w \geq 0, \end{array} \tag{4}$$

  where $w \geq 0$ means that all components of the vector are non-negative, $w_i \geq 0$, $i = 1, n$.
- The objective function of (4) is the variance of portfolio returns and R_* is the lower bound on the expected performance.

- Similarly, the optimization problem behind the *second formulation* of the principle is

$$\begin{array}{rl} \max\limits_{w} & w'\mu \\ \text{subject to} & w'e = 1 \\ & w'\Sigma w \leq R^* \\ & w \geq 0, \end{array} \tag{5}$$

in which R^* is the upper bound on the variance of the portfolio return $\sigma^2_{r_p}$.

Mean-variance optimization problems

We illustrate the two optimization problems with the following example.

- Consider three common stocks with expected returns $\mu' = (1.8\%, 2.5\%, 1\%)$ and covariance matrix,

$$\Sigma = \begin{pmatrix} 1.68 & 0.34 & 0.38 \\ 0.34 & 3.09 & -1.59 \\ 0.38 & -1.59 & 1.54 \end{pmatrix}.$$

- The variance of portfolio return equals

$$\begin{aligned} \sigma_{r_p} &= (w_1, w_2, w_3) \begin{pmatrix} 1.68 & 0.34 & 0.38 \\ 0.34 & 3.09 & -1.59 \\ 0.38 & -1.59 & 1.54 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \\ &= 1.08w_1^2 + 3.09w_2^2 + 1.54w_3^2 + 2 \times 0.34w_1w_2 \\ &- 2 \times 1.59w_2w_3 + 2 \times 0.38w_1w_3 \end{aligned}$$

and the expected portfolio return is given by

$$w'\mu = 0.018w_1 + 0.025w_2 + 0.01w_3.$$

Mean-variance optimization problems

- Correlations, which are essentially scaled covariances, are a more useful concept to see the dependence between the stocks.
- The correlation ρ_{ij} between the random return of the i-th and the j-th asset are computed by dividing the corresponding covariance by the product of the standard deviations of the two random returns,

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}}.$$

- The correlation is always bounded in the interval $[-1, 1]$.
- The closer it is to the boundaries, the stronger the dependence between the two random variables.
- If $\rho_{ij} = 1$, then the random variables are positively linearly dependent (i.e., $X_i = aX_j + b$, $a > 0$); if $\rho_{ij} = -1$, they are negatively linearly dependent (i.e., $X_i = aX_j + b$, $a < 0$).
- If the two random variables are independent, then the covariance between them is zero and so is the correlation.

- The correlation matrix ρ corresponding to the covariance matrix in this example is

$$\rho = \begin{pmatrix} 1 & 0.15 & 0.23 \\ 0.15 & 1 & -0.72 \\ 0.23 & -0.72 & 1 \end{pmatrix}.$$

- The correlation between the third and the second stock return (ρ_{32}) is -0.72, which is a strong negative correlation.
- If we observe a positive return on the second stock, it is very likely that the return on the third stock will be negative.
- We can expect that an investment split between the second and the third stock will result in a diversified portfolio.

Mean-variance optimization problems

- Suppose that we choose the expected return of the first stock ($\mu_1 = 0.018$) for the lower bound R_*.
- Optimization problem (4) has the following form,

$$
\begin{array}{ll}
\min\limits_{w_1,w_2,w_3} & \left(\begin{array}{l} 1.08w_1^2 + 3.09w_2^2 + 1.54w_3^2 + 2 \times 0.34w_1w_2 \\ -2 \times 1.59w_2w_3 + 2 \times 0.38w_1w_3 \end{array} \right) \\
\text{subject to} & w_1 + w_2 + w_3 = 1 \\
& 0.018w_1 + 0.025w_2 + 0.01w_3 \geq 0.018 \\
& w_1, w_2, w_3 \geq 0.
\end{array}
\tag{6}
$$

- Solving this problem, we obtain the optimal solution $\widetilde{w}_1 = 0.046$, $\widetilde{w}_2 = 0.509$, and $\widetilde{w}_3 = 0.445$.
- The expected return of the optimal portfolio equals $\widetilde{w}'\mu = 0.018$ and the variance of the optimal portfolio return equals $\widetilde{w}'\Sigma\widetilde{w} = 0.422$.
- There is another feasible portfolio with the same expected return and this is the portfolio composed of only the first stock.
- The variance of the return of the first stock is represented by the first element of the covariance matrix, $\sigma_{11} = 1.68$.
- If we compare the optimal portfolio $\widetilde{w}$ and the portfolio composed of the first stock only, we notice that the variance of the return of $\widetilde{w}$ is about four times below σ_{11} which means that the optimal portfolio $\widetilde{w}$ is much more diversified.

Mean-variance optimization problems

- In a similar way, we consider problem (5). Suppose that we choose the variance of the return of the first stock $\sigma_{11} = 1.68$ for the upper bound R^*. Then, the optimization problem becomes

$$\begin{array}{ll} \max\limits_{w_1,w_2,w_3} & 0.018w_1 + 0.025w_2 + 0.01w_3 \\ \text{subject to} & w_1 + w_2 + w_3 = 1 \\ & \quad 1.08w_1^2 + 3.09w_2^2 + 1.54w_3^2 + 2 \times 0.34w_1w_2 \\ & \quad -2 \times 1.59w_2w_3 + 2 \times 0.38w_1w_3 \leq 1.68 \\ & w_1, w_2, w_3 \geq 0. \end{array} \tag{7}$$

- The solution to this problem is the portfolio with weights $\widetilde{w}_1 = 0.282$, $\widetilde{w}_2 = 0.69$, and $\widetilde{w}_3 = 0.028$.
- The expected return of the optimal portfolio equals $\widetilde{w}'\mu = 0.0226$ and the variance of the optimal portfolio return equals $\widetilde{w}'\Sigma\widetilde{w} = 1.68$.
- Therefore, the optimal portfolio has the same diversification level, as indicated by variance, but it has a higher expected performance.

The mean-variance efficient frontier

- We continue the analysis by describing the set of all optimal portfolios known as the mean-variance efficient portfolios.
- Consider problem (4) and suppose that we solve it without any constraint on the expected performance.
- Then we obtain the global minimum variance portfolio. It will be the most diversified portfolio but it will have the lowest expected performance.
- The portfolio with the highest expected performance also has the highest concentration. It is composed of only one asset and this is the asset with the highest expected performance.

The mean-variance efficient frontier

- By varying the constraint on the expected return and solving problem (4), we obtain the mean-variance efficient portfolios.
- Then we can easily determine the trade-off, known as the efficient frontier, between variance and expected performance of the optimal portfolios. This trade-off
- The efficient frontier can be obtained not only from problem (4) but also from problem (5). The difference is that we vary the upper bound on the variance and maximize the expected performance.

The mean-variance efficient frontier

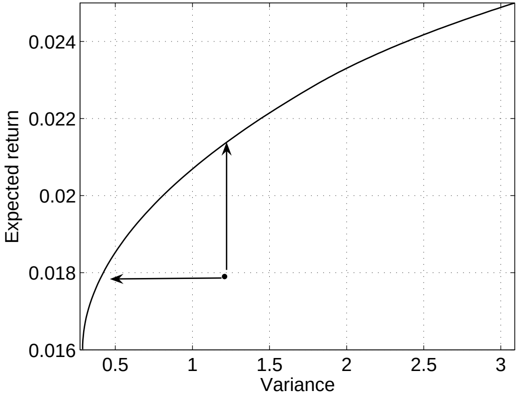

Figure 1. The plot shows the efficient frontier corresponding to the example in the previous section in the mean-variance plane. The dot indicates the position of a sub-optimal initial portfolio and the arrows indicate the position of the optimal portfolios obtained by minimizing variance or maximizing expected return.

- In Figure 1, the dot indicates the position of the portfolio with composition $w_1 = 0.8,\ w_2 = 0.1$, and $w_3 = 0.1$ in the mean-variance plane.
- It is sub-optimal as it does not belong to the mean-variance efficient portfolios. We will consider this portfolio as the initial portfolio.

The mean-variance efficient frontier

The part of the efficient frontier which contains the set of all portfolios more efficient than the initial portfolio can be obtained in the following way.

- *First*, we solve problem (4) setting the lower bound R_* equal to the expected return of the initial portfolio. The corresponding optimal solution can be found on the efficient frontier by following the horizontal arrow in *Figure 1*.
- *Second*, we solve problem (5) setting the upper bound R^* equal to the variance of the initial portfolio. The corresponding optimal solution can be found on the efficient frontier by following the vertical arrow in *Figure 1*.

The arc on the efficient frontier closed between the two arrows corresponds to the portfolios which are more efficient than the initial portfolio according to the criteria of M-V analysis — these portfolios have lower variance and higher expected performance.

The mean-variance efficient frontier

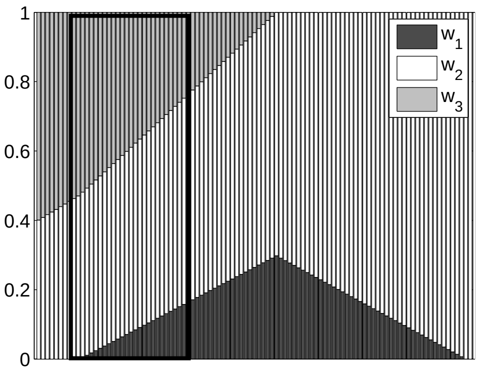

Figure 2. The plot shows the compositions of the optimal portfolios along the efficient frontier. The black rectangle indicates the portfolios more efficient than the initial portfolio.

The mean-variance efficient frontier

- In Figure 2, for each point on the efficient frontier, it shows the corresponding optimal allocation.
- For example, the optimal solution corresponding to the maximum performance portfolio consists of the second stock only. This portfolio is at the highest point of the efficient frontier and its composition is the first bar looking from right to left.
- The black rectangle shows the compositions of the more efficient portfolios than the initial portfolio. We find these by projecting the arc closed between the two arrows on the horizontal axis and then choosing the bars below it.

The mean-variance efficient frontier

- Sometimes, the efficient frontier is shown with standard deviation instead of variance on the horizontal axis.
- The set of mean-variance efficient portfolios remains unchanged because it does not matter whether we minimize the variance or the standard deviation of portfolio return as any of the two can be derived from the other by means of a monotonic function.
- Only the shape of the efficient frontier changes since we plot the expected return against a different quantity. In fact, in illustrating notions such as the capital market line or the Sharpe ratio, it is better if standard deviation is employed.

Mean-variance analysis and SSD

- A venture dominates another venture according to second order stochastic dominance (SSD) if all non-satiable, risk-averse investors prefer it.
- Suppose that a portfolio with composition $w = (w_1, \ldots, w_n)$ dominates another portfolio $v = (v_1, \ldots, v_n)$ according to SSD on the space of returns.
- *Is it true that M-V analysis will identify the portfolio v as not more efficient than w?*
- It seems reasonable to expect that such a consistency should hold,

$$w'X \succeq_{SSD} v'X \quad \Longrightarrow \quad \begin{cases} v'\mu \leq w'\mu \\ v'\Sigma v \geq w'\Sigma w. \end{cases}$$

- However, the consistency question has, generally, a negative answer.
- It is only under specific conditions concerning the multivariate distribution of the random returns X that such a consistency exits.
- Thus, the behavior of an investor making decisions according to M-V analysis is not in keeping with the class of non-satiable, risk-averse investors.
- It is possible to identify a group of investors the behavior of which is consistent with M-V analysis. This is the class of investors with quadratic utility functions,

$$u(x) = ax^2 + bx + c, \; x \in \mathbb{R}.$$

Mean-variance analysis and SSD

- Denote the set of quadratic utility functions by $\mathcal{Q}$.
- If a portfolio is not preferred to another portfolio by all investors with quadratic utility functions, then M-V analysis is capable of identifying the more efficient portfolio,

$$Eu(w'X) \geq Eu(v'X),\ \forall u \in \mathcal{Q} \qquad \Longrightarrow \qquad \begin{cases} v'\mu \leq w'\mu \\ v'\Sigma v \geq w'\Sigma w. \end{cases}$$

- The consistency with investors having utility functions in $\mathcal{Q}$ arises from the fact that, besides the basic principle in M-V analysis, there is another way to arrive at the mean-variance efficient portfolios.
- There is an optimization problem which is equivalent to problems (4) and (5). This problem is

$$\begin{array}{rl} \max\limits_{w} & w'\mu - \lambda w'\Sigma w \\ \text{subject to} & w'e = 1 \\ & w \geq 0, \end{array} \tag{8}$$

where $\lambda \geq 0$ is a parameter called the risk aversion parameter.

Mean-variance analysis and SSD

- By varying the risk aversion parameter and solving the optimization problem, we obtain the mean-variance efficient portfolios.
- For example, if $\lambda = 0$, then we obtain the portfolio with maximum expected performance.
- If the risk aversion parameter is a very large positive number, then the relative importance of the variance $w'\Sigma w$ in the objective function becomes much greater than the expected return.
- As a result, it becomes much more significant to minimize the variance than to maximize return and we obtain a portfolio which is very close to the global minimum variance portfolio.

Mean-variance analysis and SSD

- The objective function in problem (8) with λ fixed is in fact the expected utility of an investor with a quadratic utility function,

$$\begin{aligned} w'\mu - \lambda w'\Sigma w &= E(w'X) - \lambda E(w'X - E(w'X))^2 \\ &= -\lambda E(w'X)^2 + E(w'X) + \lambda(E(w'X))^2 \\ &= E(-\lambda(w'X)^2 + w'X + \lambda(E(w'X))^2) \\ &= Eg(w'X) \end{aligned}$$

  where the utility function $g(x) = -\lambda x^2 + x + \lambda b$ with b equal to the squared expected portfolio return, $b = (E(w'X))^2$.

- Since the mean-variance efficient portfolios can be obtained through maximizing quadratic expected utilities, it follows that none of these efficient portfolios can be dominated with respect to the stochastic order of quadratic utility functions.

Mean-variance analysis and SSD

- The fact that M-V analysis is consistent with the stochastic order arising from quadratic utilities, or, alternatively, it is consistent with SSD under restrictions on the multivariate distribution, means that the practical application of problems (4), (5), and (8) is limited.
- Nevertheless, sometimes quadratic approximations to more general utility functions may be sufficiently accurate, or under certain conditions the multivariate normal distribution may be a good approximation for the multivariate distribution of asset returns.

Adding a risk-free asset

- If we add a risk-free asset to the investment universe, the efficient frontier changes, the efficient portfolios becomes superior.
- The efficient portfolios essentially consist of a combination of a particular portfolio of the risky assets called the market portfolio and the risk-free asset.

Adding a risk-free asset

- Suppose that in addition to the risky assets in the investment universe, there is a risk-free asset with return r_f. The investor can choose between the n risky asset and the risk-free one.
- The weight corresponding to the risk-free asset we denote by w_f which can be positive or negative if we allow for borrowing or lending at the risk-free rate.
- We keep the notation $w = (w_1, \ldots, w_n)$ for the vector of weights corresponding to the risky assets. If we include the risk-free asset in the portfolio, the expected portfolio return equals

$$Er_p = w'\mu + w_f r_f$$

and the expression for portfolio variance remains unchanged because the risk-free asset has zero variance and therefore does not appear in the expression,

$$\sigma_{r_p}^2 = w'\Sigma w.$$

Adding a risk-free asset

- As a result, problem (4) transforms into

$$\begin{array}{rl} \min\limits_{w,w_f} & w'\Sigma w \\ \text{subject to} & w'e + w_f = 1 \\ & w'\mu + w_f r_f \geq R_* \\ & w \geq 0,\ w_f \leq 1 \end{array} \tag{9}$$

  and the equivalent problems 5) and (8) change accordingly.
- The new set of mean-variance efficient portfolios is obtained by varying the lower bound on the expected performance R_*.
- The optimal portfolios of problem (9) are always a combination of one portfolio of the risky assets and the risk-free asset.
- Changing the lower bound R_* results in different relative proportions of the two.

Adding a risk-free asset

- The portfolio of the risky assets is known as the market portfolio and is denoted by $w_M = (w_{M1}, \ldots, w_{Mn})$, the weights sum up to 1.
- All efficient portfolios can be represented as

$$\begin{aligned} r_p &= (aw_M)'X + (1 - (aw_M)'e)r_f \\ &= (aw_M)'X + (1 - a)r_f \end{aligned} \tag{10}$$

  where aw_M denotes the scaled weights of the market portfolio, a is the scaling coefficient, $1 - a = r_f$ is the weight of the risk-free asset, and we have used that $w_M'e = 1$.
- The market portfolio is located on the efficient frontier, where a straight line passing through the location of the risk-free asset is tangent to the efficient frontier. The straight line is known as the capital market line and the market portfolio is also known as the tangency portfolio.

Adding a risk-free asset

Figure below shows the efficient frontier of the example in the previous section but with standard deviation instead of variance on the horizontal axis.

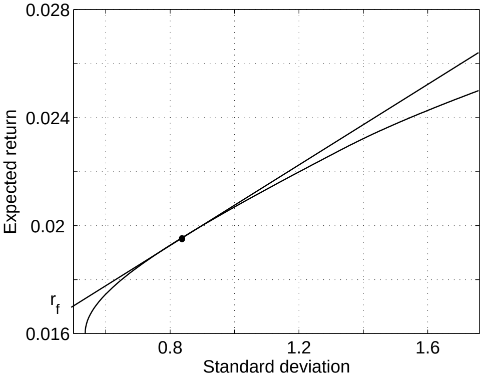

Figure 3. The dot indicates the position of the market portfolio, where the capital market line is tangent to the efficient frontier. The risk-free rate r_f is shown on the vertical axis and the straight line is the capital market line.

Adding a risk-free asset

It is possible to derive the equation of the capital market line.

- Using equation (10), the expected return of an efficient portfolio set equals,

$$\begin{aligned} E(r_p) &= aE(r_M) + (1-a)r_f \\ &= r_f + a(E(r_M) - r_f), \end{aligned}$$

  where $r_M = w_M' X$ equals the return of the market portfolio.

- The scaling coefficient a can be expressed by means of the standard deviation.

Adding a risk-free asset

- The second term in equation (10) is not random and therefore the standard deviation σ_{r_p} equals

$$\sigma_{r_p} = a\sigma_{r_M}.$$

- As a result, we derive the capital market line equation

$$E(r_p) = r_f + \left(\frac{E(r_M) - r_f}{\sigma_{r_M}}\right)\sigma_{r_p} \tag{11}$$

which describes the efficient frontier with the risk-free asset added to the investment universe.

Adding a risk-free asset

- Since any efficient portfolio is a combination of two portfolios, equation (10) is sometimes referred to as two-fund separation.
- We remark that a fund separation result such as (10) may not hold in general.
- It holds under the constraints in problem (9) but may fail if additional constraints on the portfolio weights are added.

Mean-risk analysis

- The key concept behind M-V analysis is diversification and in order to measure the degree of diversification variance, or standard deviation, is employed.
- The main idea of Markowitz is that the optimal trade-off between risk and return should be the basis of financial decision-making. The standard deviation of portfolio returns can only be used as a proxy for risk as it is not a true risk measure but a measure of dispersion.
- If we employ a true risk measure and then study the optimal trade-off between risk and return, we obtain an extension of the framework of M-V analysis which we call mean-risk analysis (M-R analysis).

Mean-risk analysis

The main principle of M-R analysis can be formulated in a similar way to M-V analysis:

1. From all feasible portfolios with a given lower bound on the expected performance, find the ones that have minimum risk.
2. From all feasible portfolios with a given upper bound on risk, find the ones that have maximum expected performance.

A key input to M-R analysis is the particular risk measure we would like to employ. The risk measure is denoted by $\rho(X)$ where X is a random variable describing portfolio return.

Mean-risk optimization problems

We can formulate two optimization problems on the basis of the main principle of M-R analysis.

- The optimization problem behind the first formulation of the principle is

$$
\begin{array}{rl}
\min\limits_{w} & \rho(r_p) \\
\text{subject to} & w'e = 1 \\
& w'\mu \geq R_* \\
& w \geq 0,
\end{array}
\tag{12}
$$

  The objective function of (4) is the risk of portfolio return $r_p = w'X$ as computed by the selected risk measure ρ and R_* is the lower bound on the expected portfolio return.

- Similarly, the the optimization problem behind the second formulation of the principle is

$$\begin{array}{rl} \max\limits_{w} & w'\mu \\ \text{subject to} & w'e = 1 \\ & \rho(r_p) \leq R^* \\ & w \geq 0, \end{array} \tag{13}$$

where R^* is the upper bound on portfolio risk.

Mean-risk optimization problems

Mean-risk optimization problems are different from their counterparts in M-V analysis.

- In order to calculate the risk of the portfolio return $\rho(r_p)$, we need to know the multivariate distribution of the asset returns.
- Otherwise, it will not be possible to calculate the distribution of the portfolio return and, as a result, portfolio risk will be unknown.
- This requirement is not so obvious in the mean-variance optimization problems where we only need the covariance matrix as input. M-V analysis leads to reasonable decision-making only under certain distributional hypotheses such as the multivariate normal distribution.

- The principal difference between mean-risk and mean-variance optimization problems is that the risk measure ρ may capture completely different characteristics of the portfolio return distribution.
- We illustrate problems (12) and (13) when the average value-at-risk (AVaR) is selected as a risk measure.
- By definition, AVaR at tail probability ϵ, $AVaR_\epsilon(X)$, is the average of the value-at-risk (VaR) numbers larger than the VaR at tail probability ϵ.
- Substituting $AVaR_\epsilon(X)$ for $\rho(X)$ in (12) and (13), we obtain the corresponding AVaR optimization problems.

Mean-risk optimization problems

- The choice of AVaR as a risk measure allows certain simplifications of the optimization problems.
- If there are available scenarios for assets returns, we can use the equivalent AVaR definition in equation **(2)** and construct problem **(8)** in *Lecture 7* and substitute problem **(8)** for the risk measure ρ.

Mean-risk optimization problems

- Denote the scenarios for the assets returns by $r^1, r^2, \ldots, r^k$ where r^j is a vector of observations,

$$r^j = (r_1^j, r_2^j, \ldots, r_n^j),$$

which contains the returns of all assets observed in a given time instant denoted by the index j.

- All observations can be arranged in a $k \times n$ matrix,

$$H = \begin{pmatrix} r_1^1 & r_2^1 & \ldots & r_n^1 \\ r_1^2 & r_2^2 & \ldots & r_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ r_1^k & r_2^k & \ldots & r_n^k \end{pmatrix}, \qquad (14)$$

in which the rows contain assets returns observed in a given moment and the columns contain all observations for one asset in the entire time period.

- The notation $r^1, r^2, \ldots, r^k$ stands for the corresponding rows of the matrix of observations H.
- We remark that the matrix H may not only be a matrix of observed returns.
- For example, it can be a matrix of independent and identically distributed scenarios produced by a multivariate model.
- In this case, k denotes the number of multivariate scenarios produced by the model and n denotes the dimension of the random vector. In contrast, if H contains historical data, then k is the number of time instants observed and n is the number of assets observed.

Mean-risk optimization problems

- Problem **(8)** contains one-dimensional observations on a random variable which, in our case, describes the return of a given portfolio.
- Therefore, the observed returns of a portfolio with composition w are $r^1 w, r^2 w, \ldots, r^k w$, or simply as the product Hw of the historical data matrix H and the vector-column of portfolio weights w.
- We restate problem **(8)** employing matrix notation,

$$\begin{array}{rcl} AVaR_{\epsilon}(Hw) & = \min\limits_{\theta,d} & \theta + \dfrac{1}{k\epsilon} d'e \\ & \text{subject to} & -Hw - \theta e \leq d \\ & & d \geq 0,\ \theta \in \mathbb{R} \end{array} \tag{15}$$

where $d' = (d_1, \ldots, d_k)$ is a vector of auxiliary variables, $e = (1, \ldots, 1)$, $e \in \mathbb{R}^k$ is a vector of ones, and $\theta \in \mathbb{R}$ is the additional parameter coming from the minimization formula given in equation **(2)** from *Lecture 7*.

- The first inequality in (15) concerns vectors and is to be interpreted in a component-by-component manner,

$$
-Hw - \theta e \leq d \qquad \Longleftrightarrow \qquad \left|
\begin{array}{l}
-r^1 w - \theta \leq d_1 \\
-r^2 w - \theta \leq d_2 \\
\dots \\
-r^k w - \theta \leq d_k
\end{array}
\right.
$$

There are very efficient algorithms for solving problems of type (12) called linear programming problems.

- Our goal is to obtain a more simplified version of problem (12) in which we minimize portfolio AVaR by changing the portfolio composition w.
- Employing (15) to calculate AVaR, we have to perform an additional minimization with respect to w and add all constraints existing in problem (12). The resulting optimization problem is

$$\begin{array}{rl} \min\limits_{w,\theta,d} & \theta + \dfrac{1}{k\epsilon} d'e \\ \text{subject to} & -Hw - \theta e \leq d \\ & w'e = 1 \\ & w'\mu \geq R_* \\ & w \geq 0,\ d \geq 0,\ \theta \in \mathbb{R}. \end{array} \tag{16}$$

Mean-risk optimization problems

- As a result, problem (16) has a more simple structure than (12) since the objective function is linear and all constraints are linear equalities or inequalities.
- There is a similar analogue to problem (13). It is constructed in the same way, the difference is that AVaR is in the constraint set and not in the objective function.
- For this reason, we include the objective function of (15) in the constraint set,

$$\begin{array}{rl} \max\limits_{w,\theta,d} & w'\mu \\ \text{subject to} & -Hw - \theta e \leq d \\ & w'e = 1 \\ & \theta + \frac{1}{k\epsilon} d'e \leq R^* \\ & w \geq 0,\, d \geq 0,\, \theta \in \mathbb{R}. \end{array} \tag{17}$$

Mean-risk optimization problems

- The structure of the resulting problem (17) is more simple than the one of (13) and is a linear programming problem.
- The method of combining (15) with (12) and (13) may seem artificial and not quite convincing that, for example, the solution of (17) and (13) with $\rho(r_p) = AVaR_\epsilon(Hw)$ will coincide.
- However, it can be formally proved that the solutions coincide.

The mean-risk efficient frontier

- Problems (12) and (13) are the main problems illustrating the principle behind M-R analysis.
- Varying the lower bound on expected return R_* in (12) or the upper bound on portfolio risk R^* in (13), we obtain the set of efficient portfolios.
- In a similar way to M-V analysis, plotting the expected return and the risk of the efficient portfolios in the mean-risk plane, we arrive at the mean-risk efficient frontier. It shows the trade-off between risk and expected return of the mean-risk efficient portfolios.

The mean-risk efficient frontier

We illustrate the mean-risk efficient frontier with the following example.

- Suppose that we choose AVaR as a risk measure and the investment universe consists of three stocks in the S&P 500 index:
  - Sun Microsystems Inc with weight w_1,
  - Oracle Corp with weight w_2,
  - Microsoft Corp with weight w_3.
- We use the observed daily returns in the period from December 31, 2002 to December 31, 2003.
- The historical data matrix H in equation (14) has three columns and 250 rows.

The mean-risk efficient frontier

- Since there are only 250 observations, we choose 40% for the tail probability ϵ in order to have a higher stability of the AVaR estimate from the sample.
- This means that the risk measure equals the average of the VaRs larger than the VaR at 40% tail probability which approximately equals the average loss provided that the loss is larger than the VaR at 40% tail probability.
- The expected daily returns are computed as the sample average and equal $\mu_1 = 0.17\%$, $\mu_2 = 0.09\%$, and $\mu_2 = 0.03\%$ where the indexing is consistent with the weight indexes.

The mean-risk efficient frontier

The efficient frontier is shown below.

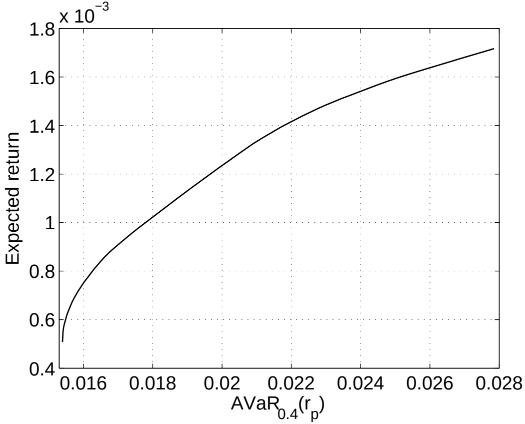

Figure 4. The plot shows the efficient frontier in the mean-risk plane. The horizontal axis ranges from about 1.5% to about 2.8%. Thus, the AVaR at 40% tail probability is about 1.5% for the global minimum risk portfolio and about 2.8% for the maximum expected return portfolio.

The mean-risk efficient frontier

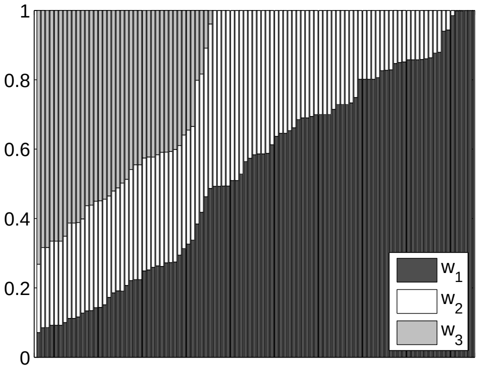

Figure 5. The plot shows the compositions of the optimal portfolios along the efficient frontier. The weight of Sun Microsystems Inc gradually increases as we move from the global minimum risk portfolio to the maximum expected return portfolio, because this stock has the highest expected daily return, $\mu_1 = 0.17\%$.

The mean-risk efficient frontier

The plot below shows the same efficient frontier as in *Figure 4* and dots indicate the positions of the three portfolios in the mean-risk plane.

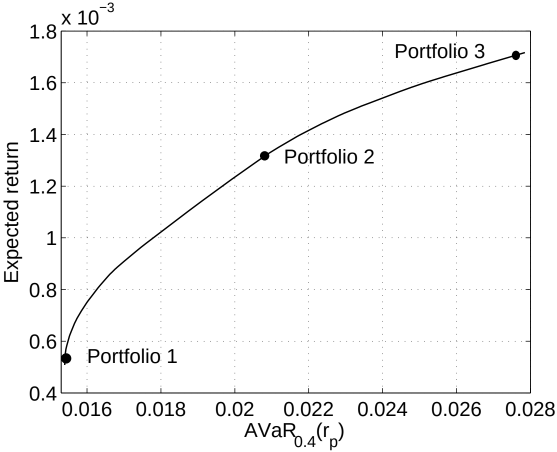

Figure 6. The plot shows the efficient frontier with three portfolios selected.

The mean-risk efficient frontier

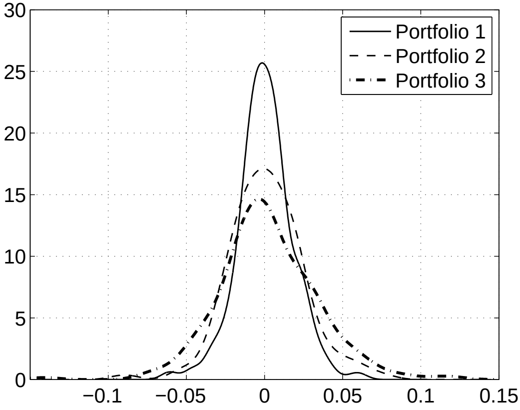

Figure 7. The plot shows the densities of the three portfolios computed from the empirical data.

The mean-risk efficient frontier

- Portfolio 1 is the global minimum risk portfolio and its density is very concentrated about the portfolio expected return.
- Portfolio 2 is in the middle part of the efficient frontier. Its density is more dispersed and slightly skewed to the right.
- The density of Portfolio 3, which is close to the maximum expected return portfolio, is much more dispersed.

- Besides problems (12) and (13), there exists another, equivalent way to obtain the mean-risk efficient frontier. This approach is based on the optimization problem

$$\begin{array}{rl} \max\limits_{w} & w'\mu - \lambda\rho(r_p) \\ \text{subject to} & w'e = 1 \\ & w \geq 0, \end{array} \tag{18}$$

  where $\lambda \geq 0$ is a risk-aversion parameter.

- By varying λ and solving problem (18), we derive a set of efficient portfolios which is obtained either through (12) or (13).

The mean-risk efficient frontier

Note that the general shape of the mean-risk efficient frontier in *Figure 4* is very similar to the shape of the mean-variance efficient frontier in *Figure 1*.

- Both are increasing functions; that is, the more risk we are ready to undertake, the higher the expected portfolio return.
- Both efficient frontiers have a concave shape; that is, the expected portfolio return gained by undertaking one additional unit of risk decreases.
- The efficient frontiers are very steep at the global minimum risk portfolio and become more flat close to the maximum expected return portfolio.

- The common properties of the frontiers on Figure 1 and Figure 4 are not accidental.
- They are governed by the properties of the risk measure $\rho(X)$, or the standard deviation in the case of M-V analysis.
- If $\rho(X)$ is convex, then the efficient frontier generated by problems (12), (13), or (18) is a concave, monotonically increasing function.
- If $\rho(X)$ belongs to the class of coherent risk measures, for example, then it is convex and, therefore, the corresponding efficient frontier has a general shape such as the one in *Figure 4*.

Mean-risk analysis and SSD

The question of consistency with SSD arises for M-R analysis as well.

- Suppose that non-satiable, risk-averse investors do not prefer a portfolio with composition $v = (v_1, \ldots, v_n)$ to another portfolio with composition $w = (w_1, \ldots, w_n)$.
- If X is a random vector describing the returns of the assets in the two portfolios, then is M-R analysis capable of indicating that the portfolio with return $v'X$ is not less efficient than $w'X$?
- A reasonable consistency condition is the following one

$$w'X \succeq_{SSD} v'X \quad \Longrightarrow \quad \begin{cases} v'\mu \leq w'\mu \\ \rho(v'X) \geq \rho(w'X). \end{cases} \tag{19}$$

- It is the risk measure $\rho(X)$ which should be endowed with certain properties in order for (19) to hold true.

Mean-risk analysis and SSD

- If $\rho(X)$ is a coherent risk measure, then it does not necessarily follow that (19) will hold.
- For some particular representatives, the consistency condition is true. For instance, if $\rho(X)$ is AVaR or, more generally, a spectral risk measure, then it is consistent with SSD.
- Since AVaR is consistent with SSD, the set of efficient portfolios, generated for instance by problem (12) with $\rho(X) = AVaR_\epsilon(X)$, does not contain a pair of two portfolios w and v such that all non-satiable, risk-averse investors prefer strictly one to the other, $w'X \succ_{SSD} v'X$.

Mean-risk analysis and SSD

To verify the previous statement, assume the converse.

- If $w'X$ dominates strictly $v'X$ according to SSD, then one of the inequalities in (19) is strict.
- The portfolio v cannot be a solution to the optimization problems generating the efficient frontier, which results in a contradiction to the initial assumption.
- The conclusion is that none of the efficient portfolios can dominate strictly another efficient portfolio with respect to SSD.
- Therefore, which portfolio on the efficient frontier an investor would choose depends entirely on the particular functional form of the investor's utility function.
- If the investor is very risk-averse, then the optimal choice will be a portfolio close to the globally minimum risk portfolio and if the investor is risk-loving, then a portfolio close to the other end of the efficient frontier may be preferred.

Risk versus dispersion measures

- The global minimum risk portfolio can be calculated from problem (12) by removing the lower bound on the expected portfolio return.
- In this way, we solve a problem without any requirements on the expected performance. Even though we remove the constraint, the expected portfolio return may still influence the optimal solution.
- Suppose that $\rho(X)$ is a coherent risk measure. Then, changing only the expectation of the portfolio return distribution by adding a positive constant results in a decrease of risk,

$$\rho(X + C) = \rho(X) - C$$

where C is a positive constant.
- Then any coherent risk measure can be represented as

$$\rho(X) = \rho(X - EX) - EX. \tag{20}$$

- The first term in the difference is completely independent of the expected value of X.
- As a result of this decomposition, problem (12) can be restated without the expected return constraint in the following way,

$$\begin{array}{rl} \max\limits_{w} & w'\mu - \rho(r_p - w'\mu) \\ \text{subject to} & w'e = 1 \\ & w \geq 0, \end{array} \tag{21}$$

  where we have changed the minimization to maximization and have flipped the sign of the objective function.
- The solution to problem (21) is the global minimum risk portfolio and the expected portfolio return $w'\mu$ has a certain impact on the solution as it appears in the objective function.

Risk versus dispersion measures

- In contrast, the global minimum variance portfolio in M-V analysis does not share this property.
- It is completely invariant of the expected returns of the assets in the investment universe.
- This difference between M-R analysis and M-V analysis is not to be regarded as a drawback of one or the other. It is one consequence of employing a risk measure in the optimization problem.
- In spite of the differences between the two, under certain conditions it appears possible to extend the mean-risk efficient frontier by substituting the risk measure for a suitable dispersion measure so that the mean-risk efficient frontier properties become more similar to the properties of the mean-variance efficient frontier.

Risk versus dispersion measures

There exists a connection between a sub-family of the coherent risk measures and a family of dispersion measures.

- Suppose that $\rho(X)$ is a coherent risk measure and, additionally, it satisfies the property $\rho(X) > -EX$.
- Suppose that $D(X)$ is a deviation measure and, additionally, it satisfies the property $D(X) \leq EX$ for all non-negative random variables, $X \geq 0$.
- Under these assumptions, any of the two functionals can be expressed from the other in the following way,

$$D(X) = \rho(X - EX)$$
$$\rho(X) = D(X) - EX.$$

$\Rightarrow$ Here we'll always assume that $D(X)$ and $\rho(X)$ are such that the relationship above holds.

Risk versus dispersion measures

- Consider the objective function of problem (18). Applying the decomposition in equation (20), we obtain

$$
\begin{aligned}
w'\mu - \lambda\rho(r_p) &= w'\mu - \lambda\rho(r_p - w'\mu) + \lambda w'\mu \\
&= (1+\lambda)w'\mu - \lambda\rho(r_p - w'\mu) \\
&= (1+\lambda)\left(w'\mu - \frac{\lambda}{1+\lambda}\rho(r_p - w'\mu)\right).
\end{aligned}
$$

- Since $\lambda \geq 0$, we can safely ignore the positive factor $1+\lambda$ in the objective function because it does not change the optimal solution.

- In effect, we obtain the following optimization problem, which is equivalent to (18),

$$\begin{array}{ll} \max\limits_{w} & w'\mu - \dfrac{\lambda}{1+\lambda}\rho(r_p - w'\mu) \\ \text{subject to} & w'e = 1 \\ & w \geq 0. \end{array} \tag{22}$$

Risk versus dispersion measures

- We recognize the deviation measure $D(r_p) = \rho(r_p - w'\mu)$ in the objective function.
- Note that the aversion coefficient is not an arbitrary positive number, $\lambda/(1+\lambda) \in [0,1]$, because of the assumption that the risk-aversion coefficient is non-negative.
- As a result, we can see the parallel between (22) and the corresponding problem with a deviation measure,

$$\begin{array}{rl} \max\limits_{w} & w'\mu - cD(r_p) \\ \text{subject to} & w'e = 1 \\ & w \geq 0, \end{array} \tag{23}$$

where $c \geq 0$ is the corresponding aversion coefficient.

Risk versus dispersion measures

- The set of optimal portfolios obtained from (23) by varying the parameter c contains the set of mean-risk efficient portfolios of (22).
- The efficient frontier corresponding to (23) has properties similar to the mean-variance efficient frontier since $D(r_p)$ does not depend on the expected portfolio return.
- The optimal portfolios, which appear in addition to the mean-risk efficient portfolios, are obtained with $c > 1$.
- If $c < 1$, then there is an equivalent $\lambda = c/(1 - c)$ such that the optimal portfolios of (22) coincide with the optimal solutions of (23).
- Increasing c, we obtain more and more diversified portfolios. In effect, the left part of the mean-risk efficient frontier gets extended by problem (23).

Risk versus dispersion measures

- Actually, in the mean-risk plane, the extended part curves back because these portfolios are sub-optimal according to M-R analysis while in mean-deviation plane, the efficient frontier is a concave, monotonically increasing function.
- The difference between the mean-risk and the mean-deviation planes is merely a change in coordinates given by equation (20).

Risk versus dispersion measures

- The set of optimal portfolios additional to the mean-risk efficient portfolios can be large or small depending on the magnitude of the expected returns of the assets.
- If the expected returns are close to zero, the set is small and it completely disappears if the expected returns are exactly equal to zero.
- In practice, if we use daily returns, the efficient portfolios generated by (22) and (23) almost coincide. Larger discrepancies may appear with weekly or monthly data.

- In order to see the usual magnitude of the extension of the mean-risk efficient portfolios by (23), we increase five times the expected returns of the common stocks in the example developed in Section "The mean-risk efficient frontier" keeping everything else unchanged.
- The increase roughly corresponds to the magnitude of weekly expected returns.
- The resulting mean-risk efficient frontier and set of efficient portfolios is given in *Figures 8,9*.

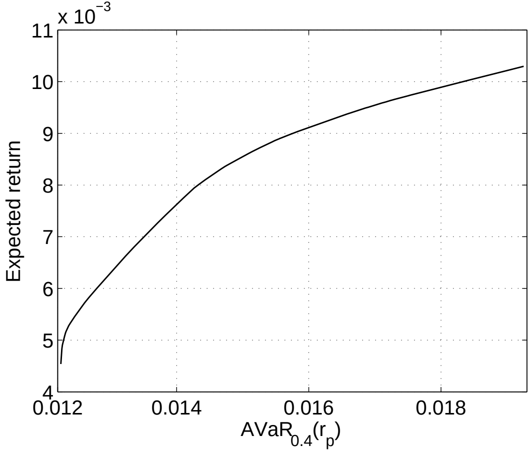

Figure 8. The plot shows the efficient frontier in the mean-risk plane.

Risk versus dispersion measures

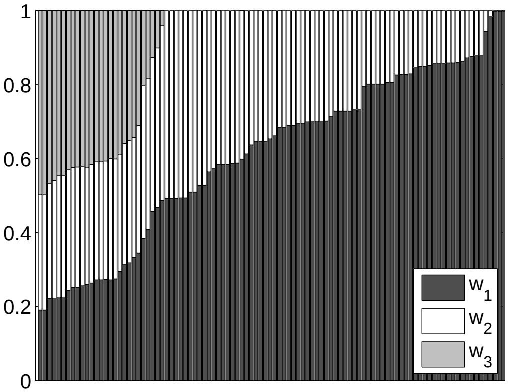

Figure 9. The plot shows the compositions of the optimal portfolios along the efficient frontier.

Risk versus dispersion measures

The efficient portfolios generated by problem (23) with $D(X) = AVaR_{0.4}(X - EX)$ are shown in *Figures 10,11*.

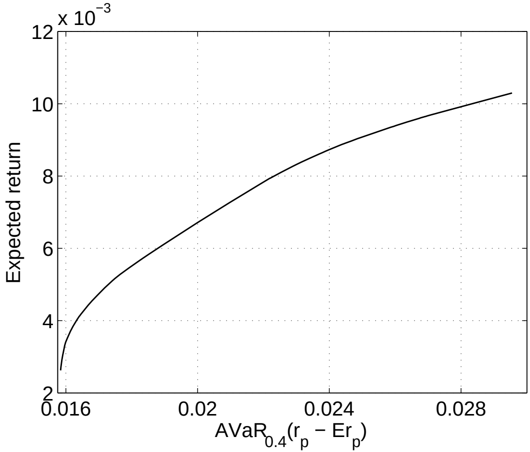

Figure 10. The plot shows the efficient portfolios coordinates of (23) in the mean-deviation plane. The rectangle indicates the portfolios additional to the

Risk versus dispersion measures

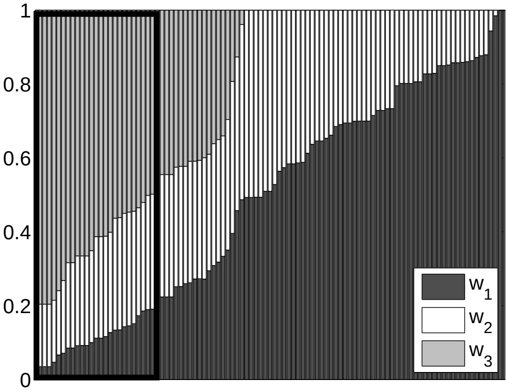

Figure 11. The plot shows the compositions of the optimal portfolios. The rectangle indicates the optimal portfolios which are additional to the mean-risk efficient portfolios.

Note the difference between the horizontal axes in *Figures* 8,9 and

- As a next step, we plot the coordinates of the additional portfolios in the mean-risk plane.
- These portfolios are sub-optimal according to M-R analysis and, therefore, the extension of the mean-risk efficient frontier will curve backwards. This is illustrated in *Figure 12*.
- Sub-optimal has an easy geometric illustration. For any of these portfolios, we can find an equally risky portfolio with a higher expected return, which is on the mean-risk efficient frontier.

Risk versus dispersion measures

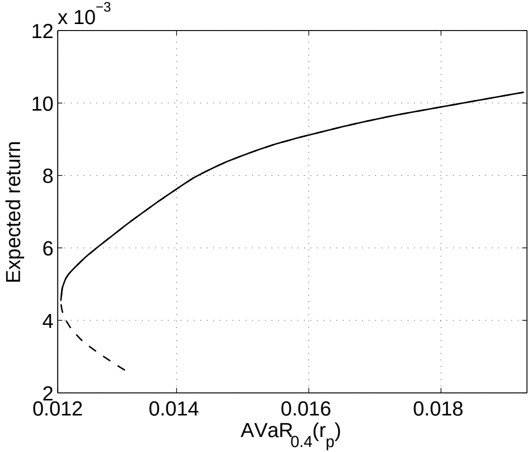

Figure 12. The mean-risk efficient frontier with the coordinates of the additional optimal portfolios plotted with a dashed line. The portfolios which are indicated by the rectangle in *Figure 11* are shown with a dashed line in the mean-risk plane here.

Risk versus dispersion measures

If M-R analysis leads to the conclusion that these portfolios are sub-optimal, why do we consider them at all?

- Suppose that we are uncertain about the reliability of the expected return estimates and we want to minimize the impact of this uncertainty on the optimal solution.
- Since the means affect the global minimum risk portfolio, we may want to reduce further the effect of the means by moving to the extension of the efficient frontier given by the mean-deviation optimization problem (23).
- The portfolio which appears at the very end of the dashed line in *Figure 12* is the minimum dispersion portfolio, the composition of which is not influenced by the means at all.
- In effect, even though the mean-deviation optimal portfolios are sub-optimal, under certain circumstances they may still be of practical interest.

Risk versus dispersion measures

- We can classify all optimal portfolios obtained from the mean-risk optimization problem of the following type

$$\begin{array}{rl} \min\limits_{w} & \rho(r_p) \\ \text{subject to} & w'e = 1 \\ & w'\mu = R_* \\ & w \geq 0. \end{array} \tag{24}$$

- The expected return constraint in (12) is an inequality and in (24) it is an equality.
- This may seem to be an insignificant modification of the initial problem but it results in problem (24) being more general than (12) in the following sense.

Risk versus dispersion measures

- The optimal portfolios obtained by varying the bound R_* in (24) contain the mean-risk efficient portfolios and, more generally, the mean-deviation efficient portfolios.
- By fixing the expected portfolio return to be equal to R_*, we are essentially minimizing portfolio dispersion.
- By equation (20), the objective function of problem (24) can be written as

$$\rho(r_p) = D(r_p) - w'\mu = D(r_p) - R_*$$

  in which R_* is a constant and, therefore, it cannot change the optimal solution. In practice, we are minimizing the dispersion $D(r_p)$.

Risk versus dispersion measures

The optimal portfolios generated by problem (24) by varying R_* can be classified into three groups. *Figure 13* illustrates the groups.

- The dark gray group contains the mean-risk efficient portfolios generated by (12), obtained from (24) with high values of R_*.
- The gray group contains the mean-deviation efficient portfolios produced by problem (23) which are not mean-risk efficient. They are obtained from (24) with medium values of R_*.
- Finally, the white set consists of optimal portfolios which are not mean-deviation efficient but solve (24). They are obtained with small values of R_*. This set has no practical significance since the portfolios belonging to it have small expected returns and high dispersions.

Risk versus dispersion measures

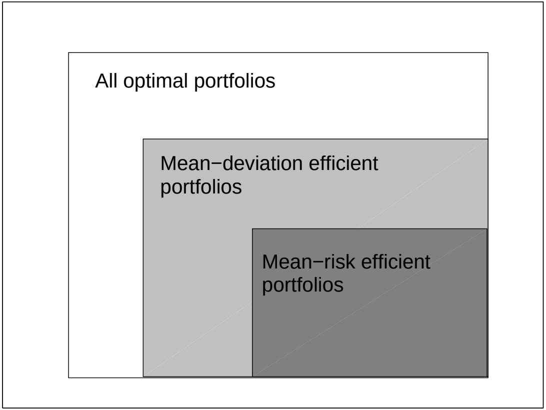

Figure 13. Classification of the optimal portfolios generated by problem (24) by varying the expected return bound R_*.

Summary

- We described M-V analysis and the associated optimal portfolio problems.
- We discussed the mean-variance efficient frontier and consistency of M-V analysis with the stochastic dominance order of the class of non-satiable, risk-averse investors.
- Considering a true risk measure instead of standard deviation leads to M-R analysis. The same reasoning leads to the mean-risk efficient frontier which, under certain conditions, is related to a mean-dispersion efficient frontier.
- As a result of this relationship, we demonstrated that all optimal portfolios can be classified into three groups — mean-risk efficient portfolios, mean-dispersion efficient portfolios which are not mean-risk efficient, and optimal portfolios which are not mean-dispersion efficient.
- In the appendix to this lecture, we remark on the numerical difficulties in solving the optimal portfolio problems when AVaR is selected as a risk measure.

Svetlozar T. Rachev, Stoyan Stoyanov, and Frank J. Fabozzi
Advanced Stochastic Models, Risk Assessment, and Portfolio Optimization: The Ideal Risk, Uncertainty, and Performance Measures
John Wiley, Finance, 2007.

Chapter 8.