Technical Appendix

Lecture 6: Risk and uncertainty

Prof. Dr. Svetlozar Rachev

Institute for Statistics and Mathematical Economics
University of Karlsruhe

Portfolio and Asset Liability Management

Summer Semester 2008

Copyright

The material is based on the text-book:

Svetlozar T. Rachev, Stoyan Stoyanov, and Frank J. Fabozzi

Advanced Stochastic Models, Risk Assessment, and Portfolio Optimization: The Ideal Risk, Uncertainty, and Performance Measures

John Wiley, Finance, 2007

Prof. Svetlozar (Zari) T. Rachev
Chair of Econometrics, Statistics
and Mathematical Finance
School of Economics and Business Engineering
University of Karlsruhe
Kollegium am Schloss, Bau II, 20.12, R210
Postfach 6980, D-76128, Karlsruhe, Germany
Tel. +49-721-608-7535, +49-721-608-2042(s)
Fax: +49-721-608-3811
http://www.statistik.uni-karslruhe.de

- The sub-additivity and the positive homogeneity properties of coherent risk measures guarantee that they are convex.
- The convexity property describes the diversification effect when the random variables are interpreted as portfolio returns.
- It is possible to postulate convexity directly and obtain the larger class of convex risk measures.

A risk measure ρ is said to be a convex risk measure if it satisfies the following properties.

Monotonicity	$\rho(Y) \leq \rho(X)$, if $Y \geq X$ in almost sure sense.
Convexity	$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$, for all X, Y and $\lambda \in [0, 1]$
Invariance	$\rho(X + C) = \rho(X) - C$, for all X and $C \in \mathbb{R}$.

- The remarks concerning the interpretation of the axioms of coherent risk measures depending on whether X describes payoff or return are valid for the convex risk measures as well.
- The convex risk measures are more general than the coherent risk measures because every coherent risk measure is convex but not vice versa.
- The convexity property does not imply positive homogeneity.
- *Föllmer* and *Schied* (2002) provide more details on convex risk measures and their relationship with preference relations.

- We demonstrate that the symmetric deviation measures arise from probability metrics equipped with two additional properties — translation invariance and positive homogeneity.
- Not only the symmetric but all deviation measures can be described with the general method of probability metrics by extending the framework [1].

[1] This is illustrated in the appendix to Lecture 9.

We briefly repeat the definition of a probability semimetric.

- The probability semimetric is denoted by $\mu(X, Y)$ in which X and Y are random variables.
- The properties which $\mu(X, Y)$ should satisfy are the following.

Property 1. $\mu(X, Y) \geq 0$ for any X, Y and $\mu(X, Y) = 0$ if $X = Y$ in almost sure sense.

Property 2. $\mu(X, Y) = \mu(Y, X)$ for any X, Y.

Property 3. $\mu(X, Y) \leq \mu(X, Z) + \mu(Z, Y)$ for any X, Y, Z.

- A probability metric is called translation invariant and positively homogeneous if, besides properties 1, 2, and 3, it satisfies also

Property 4. $\mu(X+Z, Y+Z) = \mu(Y, X)$ for any X, Y, Z.

Property 5. $\mu(aX, aY) = a\mu(X, Y)$ for any X, Y and $a > 0$.

$\Rightarrow$ Property 4 is the translation invariance axiom and Property 5 is the positive homogeneity axiom.

- Note that translation invariance and positive homogeneity have a different meaning depending on whether probability metrics or dispersion measures are concerned.
- To avoid confusion, we enumerate the axioms of symmetric deviation measures $D(X)$.

Property 1*. $D(X + C) = D(X)$ for all X and constants $C \in \mathbb{R}$.

Property 2*. $D(X) = D(-X)$ for all X.

Property 3*. $D(0) = 0$ and $D(\lambda X) = \lambda D(X)$ for all X and all $\lambda > 0$.

Property 4*. $D(X) \geq 0$ for all X, with $D(X) > 0$ for non-constant X.

Property 5*. $D(X + Y) \leq D(X) + D(Y)$ for all X and Y.

- We will demonstrate that the functional

$$\mu_D(X, Y) = D(X - Y) \tag{1}$$

is a probability semimetric satisfying properties 1 through 5 if D satisfies properties 1* through 5*.

- Furthermore, the functional

$$D_\mu(X) = \mu(X - EX, 0) \tag{2}$$

is a symmetric deviation measure if μ is a probability metric satisfying properties 2 through 5.

We show that properties 1 through 5 hold for μ_D defined in equation (1).

Property 1. $\mu_D(X,Y) \geq 0$ follows from the non-negativity of D, Property 4*. Further on, if $X = Y$ in almost sure sense, then $X - Y = 0$ in almost sure sense and $\mu_D(X,Y) = D(0) = 0$ from Property 3*.

Property 2. A direct consequence of Property 2*.

Property 3. Follows from Property 5*:

$$\begin{aligned}\mu(X,Y) &= D(X-Y) = D(X-Z+(Z-Y)) \\ &\leq D(X-Z) + D(Z-Y) = \mu(X,Z) + \mu(Z,Y)\end{aligned}$$

Property 4. A direct consequence of the definition in (1).

Property 5. Follows from Property 3*.

Technical appendix

Demonstration of equation (2)

We show that properties 1* through 5* hold for D_μ defined in equation (2).

Property 1* A direct consequence of the definition in (2).

Property 2* Follows from Property 4 and Property 2.

$$D_\mu(-X) = \mu(-X + EX, 0) = \mu(0, X - EX) = \mu(X - EX, 0) = D_\mu(X)$$

Property 3* Follows from Property 1 and Property 5. $D_\mu(0) = \mu(0,0) = 0$

$$D_\mu(\lambda X) = \lambda\mu(X - EX, 0) = \lambda D_\mu(X)$$

Property 4* Follows because μ is a probability metric. If $D_\mu(X) = 0$, then $X - EX$ is equal to zero almost surely which means that X is a constant in all states of the world.

Property 5* Arises from Property 3 and Property 4.

- Equation (2) shows that all symmetric deviation measures arise from the translation invariant, positively homogeneous probability metrics.
- Note that because of the properties of the deviation measures, μ_D is a semimetric and cannot become a metric. This is because D is not sensitive to additive shifts and this property is inherited by μ_D,

$$\mu_D(X + a, Y + b) = \mu_D(X, Y),$$

  where a and b are constants.
- In effect, $\mu_D(X, Y) = 0$ implies that the two random variables differ by a constant, $X = Y + c$ in all states of the world.

- Due to the translation invariance property, equation (2) can be equivalently re-stated as

$$D_\mu(X) = \mu(X, EX). \tag{3}$$

- It represents a very natural generic way of defining measures of dispersion.
- Starting from equation (3) and replacing the translation invariance property by the regularity property of ideal probability metrics, the sub-additivity property (Property 5*) of $D_\mu(X)$ breaks down and a property similar to the positive shift property holds instead of Property 1*,

$$D_\mu(X + C) = \mu(X + C, EX + C) \leq \mu(X, EX) = D_\mu(X)$$

  for all constants C.

$\Rightarrow$ This property is more general than the positive shift property as it holds for arbitrary constants.

Svetlozar T. Rachev, Stoyan Stoyanov, and Frank J. Fabozzi
Advanced Stochastic Models, Risk Assessment, and Portfolio Optimization: The Ideal Risk, Uncertainty, and Performance Measures
John Wiley, Finance, 2007.

Chapter 6.