Technical Appendix

Lecture 5: Choice under uncertainty

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Portfolio and Asset Liability Management

Summer Semester 2008

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The material is based on the text-book:

Svetlozar T. Rachev, Stoyan Stoyanov, and Frank J. Fabozzi
Advanced Stochastic Models, Risk Assessment, and Portfolio Optimization: The Ideal Risk, Uncertainty, and Performance Measures
John Wiley, Finance, 2007

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The axioms of choice

The axioms of choice are fundamental assumptions defining a preference order.

- $\mathcal{X}$ stands for the set of the probability distributions of the ventures also known as lotteries, and the notation $P_X \succeq P_Y$ means that the economic agent prefers P_X to P_Y or is indifferent between them.
- The notation $P_X \succ P_Y$ means that P_X is strictly preferred to P_Y.

The axioms of choice

The axioms of choice are the following:

Completeness	For all $P_X, P_Y \in \mathcal{X}$, either $P_X \succeq P_Y$ or $P_Y \succeq P_X$ or both are true, $P_X \sim P_Y$.
Transitivity	If $P_X \succeq P_Y$ and $P_Y \succeq P_Z$, then $P_X \succeq P_Z$, where P_X, P_Y and P_Z are three lotteries.
Archimedean Axiom	If $P_X, P_Y, P_Z \in \mathcal{X}$ are such that $P_X \succ P_Y \succ P_Z$, then there is an $\alpha, \beta \in (0, 1)$ such that $\alpha P_X + (1-\alpha)P_Z \succ P_Y$ and also $P_Y \succ \beta P_X + (1-\beta)P_Z$.
Independence Axiom	For all $P_X, P_Y, P_Z \in \mathcal{X}$ and any $\alpha \in [0, 1]$, $P_X \succeq P_Y$ if and only if $\alpha P_X + (1-\alpha)P_Z \succeq \alpha P_Y + (1-\alpha)P_Z$.

The axioms of choice

- The *completeness* axiom states that economic agents should always be able to compare two lotteries, e.g. two portfolios. They either prefer one or the other, or are indifferent.
- The *transitivity* axiom rules out the possibility that an investor may prefer P_X to P_Y, P_Y to P_Z, and also P_Z to P_X. It states that if the first two relations hold, then necessarily the investor should prefer P_X to P_Z.
- The *Archimedean* axiom is like a "continuity" condition. It states that given any three distributions strictly preferred to each other, we can combine the most and the least preferred distribution through an $\alpha \in (0, 1)$ such that the resulting distribution is strictly preferred to the middle distribution. Likewise, we can combine the most and the least preferred distribution through a $\beta \in (0, 1)$ so that the middle distribution is strictly preferred to the resulting distribution.
- The *independence* axiom claims that the preference between two lotteries remains unaffected if they are both combined in the same way with a third lottery.

The basic result of von Neumann-Morgenstern is that a preference relation satisfies the four axioms of choice if and only if there is a real-valued function, $U : \mathcal{X} \to \mathbb{R}$, such that:

a) U represents the preference order,

$$P_X \succeq P_Y \quad \iff \quad U(P_X) \geq U(P_Y)$$

for all $P_X, P_Y \in \mathcal{X}$.

b) U has the linear property,[1]

$$U(\alpha P_X + (1-\alpha)P_Y) = \alpha U(P_X) + (1-\alpha)U(P_Y)$$

for any $\alpha \in (0, 1)$ and $P_X, P_Y \in \mathcal{X}$.

[1] Functions satisfying this property are also called *affine*.

The axioms of choice

- Moreover, the numerical representation U is unique up to a positive linear transform. That is, if U_1 and U_2 are two functions representing one and the same preference order, then $U_2 = aU_1 + b$ where $a > 0$ and b are some coefficients.
- It turns out that the numerical representation has a very special form under some additional technical continuity conditions:

$$U(P_X) = \int_{\mathbb{R}} u(x) dF_X(x)$$

  where the function $u(x)$ is the utility function of the economic agent and $F_X(x)$ is the c.d.f. of the probability distribution P_X.
- Thus, the numerical representation of the preference order of an economic agent is the expected utility of X.
- The fact that U is known up to a positive linear transform means that the utility function of the economic agent is not determined uniquely from the preference order but is also unique up to a positive linear transform.

Stochastic dominance relations of order n

- Including additional characteristics of the investors by imposing conditions on the utility function, we end up with more refined stochastic orders. This method can be generalized in the n-th order stochastic dominance.
- Denote by $\mathcal{U}_n$ the set of all utility functions, the derivatives of which satisfy the inequalities $(-1)^{k+1}u^{(k)}(x) \geq 0,\ k = 1, 2, \ldots, n$ where $u^{(k)}(x)$ denotes the k-th derivative of $u(x)$.
- For each n, we have a set of utility functions which is a subset of $\mathcal{U}_{n-1}$,

$$\mathcal{U}_1 \subset \mathcal{U}_2 \subset \ldots \subset \mathcal{U}_n \subset \ldots$$

  The classes of investors characterized by the first-, second-, and third-order stochastic dominance are $\mathcal{U}_1$, $\mathcal{U}_2$, and $\mathcal{U}_3$.

Stochastic dominance relations of order n

- Imposing further properties on the derivatives of the utility function requires that we make more assumptions for the moments of the random variables we consider.
- We assume that the absolute moments $E|X|^k$ and $E|Y|^k$, $k = 1, \ldots, n$ of the random variables X and Y are finite.
- We say that the portfolio X dominates the portfolio Y in the sense of the n-th order stochastic dominance, $X \succeq_n Y$, if no investor with a utility function in the set $\mathcal{U}_n$ would prefer Y to X,

$$X \succeq_n Y \quad \text{if} \quad Eu(X) \geq Eu(Y), \ \forall u(x) \in \mathcal{U}_n.$$

- Thus, the first-, second-, and third-order stochastic dominance appear as special cases from the n-th order stochastic dominance with $n = 1, 2, 3$.

Stochastic dominance relations of order n

There is an equivalent way of describing the n-th order stochastic dominance in terms of the c.d.f.s of the ventures only.

- The condition is the following one,

$$X \succeq_n Y \quad \Longleftrightarrow \quad F_X^{(n)}(x) \leq F_Y^{(n)}(x), \ \forall x \in \mathbb{R} \tag{1}$$

  where $F_X^{(n)}(x)$ stands for the n-th integral of the c.d.f. of X which can be defined recursively as

$$F_X^{(n)}(x) = \int_{-\infty}^{x} F_X^{(n-1)}(t)dt.$$

- An equivalent form of the condition in (1) can be derived, which is close to the form of TSD condition (8) in the lecture,

$$X \succeq_n Y \quad \Longleftrightarrow \quad E(t-X)_+^{n-1} \leq E(t-Y)_+^{n-1}, \ \forall t \in \mathbb{R} \tag{2}$$

  where $(t-x)_+^{n-1} = \max(t-x, 0)^{n-1}$.

Stochastic dominance relations of order n

- Since in the n-th order stochastic dominance we furnish the conditions on the utility function as n increases, the following relation holds,

$$X \succeq_1 Y \quad \Longrightarrow \quad X \succeq_2 Y \quad \Longrightarrow \quad \dots \quad \Longrightarrow \quad X \succeq_n Y,$$

  which generalizes the relationship between FSD, SSD, and TSD.
- It is possible to extend the n-th order stochastic dominance to the α-order stochastic dominance in which $\alpha \geq 1$ is a real number and instead of the ordinary integrals of the c.d.f.s, fractional integrals are involved [2].

[2] See Ortobelli et al. (2007) for details

Return versus payoff and stochastic dominance

- The lotteries in von Neumann-Morgenstern theory are usually interpreted as probability distributions of payoffs. That is, the domain of the utility function $u(x)$ is the positive half-line which is interpreted as the collection of all possible outcomes in terms of dollars from a given venture.
- Assume that the payoff distribution is actually the price distribution P_t of a financial asset at a future time t. In line with the von Neumann-Morgenstern theory, the expected utility of P_t for an investor with utility function $u(x)$ is given by

$$U(P_t) = \int_0^\infty u(x) dF_{P_t}(x) \tag{3}$$

where $F_{P_t}(x) = P(P_t \leq x)$ is the c.d.f. of the random variable P_t.

Return versus payoff and stochastic dominance

- Further on, suppose that the price of the common stock at the present time is P_0. Consider the substitution $x = P_0 \exp(y)$. Under the new variable, the c.d.f. of P_t changes to

$$F_{P_t}(P_0 \exp(y)) = P(P_t \leq P_0 \exp(y)) = P\left(\log \frac{P_t}{P_0} \leq y\right)$$

  which is, in fact, the distribution function of the log-return of the financial asset $r_t = \log(P_t/P_0)$.
- The integration range changes from the positive half-line to the entire real line and equation (3) becomes

$$U(P_t) = \int_{-\infty}^{\infty} u(P_0 \exp(y)) dF_{r_t}(y). \tag{4}$$

Return versus payoff and stochastic dominance

- On the other hand, the expected utility of the log-return distribution has the form

$$U(r_t) = \int_{-\infty}^{\infty} v(y) dF_{r_t}(y) \qquad (5)$$

  where $v(y)$ is the utility function of the investor on the space of log-returns which is unique up to a positive linear transform.
- Note that $v(y)$ is defined on the entire real line as the log-return can be any real number.

Return versus payoff and stochastic dominance

- Compare equations (4) and (5). From the uniqueness of the expected utility representation, it appears that (4) is the expected utility of the log-return distribution. Therefore, the utility function $v(y)$ can be computed by means of the utility function u,

$$v(y) = a.u(P_0 \exp(y)) + b, \quad a > 0 \tag{6}$$

  in which the constants a and b appear because of the uniqueness result.

- Conversely, the utility function $u(x)$ can be expressed via v,

$$u(x) = c.v(\log(x/P_0)) + d, \quad c > 0. \tag{7}$$

- Note that the two utilities in equations (4) and (5) are identical (up to a positive linear transform), because the investor is the same. We only change the way we look at the venture.

Return versus payoff and stochastic dominance

- Because of the relationship between the functions u and v, properties imposed on the utility function u may not transfer to the function v and vice versa.
- We remark on what happens with the properties connected with the n-th order stochastic dominance.
- Suppose that the utility function $v(y)$ belongs to the set $\mathcal{U}_n$, i.e. it satisfies the conditions

$$(-1)^{k+1}v^{(k)}(y) \geq 0, \quad k = 1, 2, \ldots, n$$

  where $v^{(k)}(y)$ denotes the k-th derivative of $v(y)$.
- It turns out that the function $u(x)$ given by (7) satisfies the same properties and, therefore, it also belongs to the set $\mathcal{U}_n$. This is verified directly by differentiation.

Return versus payoff and stochastic dominance

- In the reverse direction, the statement holds only for $n = 1$. If $u \in \mathcal{U}_n$, $n > 1$, then the function v given in (6) may not belong to $\mathcal{U}_n$, $n > 1$, and we obtain a set of functions to which $\mathcal{U}_n$ is a subset.
- The n-th degree stochastic dominance, $n > 1$, on the space of payoffs implies the n-th degree stochastic dominance, $n > 1$, on the space of the corresponding log-returns but not vice versa,

$$P_t^1 \succeq_n P_t^2 \quad \Longrightarrow \quad r_t^1 \succeq_n r_t^2.$$

  where P_t^1 and P_t^2 are the payoffs of the two common stocks, for example, at time $t > 0$, and r_t^1 and r_t^2 are the corresponding log-returns for the same period.
- Note that this relationship holds if we assume that the prices of the two common stocks at the present time are equal to $P_0^1 = P_0^2 = P_0$.

Other stochastic dominance relations

There are ways of obtaining stochastic dominance relations other than the n-th order stochastic dominance which is based on certain properties of investors' utility functions.

- We borrow an example from reliability theory and adapt it for distributions describing payoffs, losses or returns.
- Consider the conditional probability

$$Q_X(t,x) = P(X > t + x | X > t). \tag{8}$$

  where $x \geq 0$ and suppose that X describes a random loss.
- Then, equation (8) calculates the probability of losing more than $t + x$ on condition that the loss is larger than t. This probability may vary depending on the level t with the additional amount of loss being fixed (x does not depend on t).

Other stochastic dominance relations

- For example, if $t_1 \leq t_2$, then the corresponding conditional probabilities may be related in the following way,

$$Q_X(t_1, x) \geq Q_X(t_2, x). \tag{9}$$

- Thus, the deeper we go into the tail, the less likely it is to lose additional x dollars provided that the loss is larger than the selected threshold.
- Conversely, if the inequality is

$$Q_X(t_1, x) \leq Q_X(t_2, x), \tag{10}$$

  then the further we go into the tail, the more likely it becomes to lose additional x dollars.
- Basically, the inequalities in (9) and (10) describe certain tail properties of the random variable X.

Other stochastic dominance relations

- Denote by $\bar{F}_X(x) = 1 - F_X(x) = P(X > x)$ the tail of the random variable X. Then, according to the definition of conditional probability, equation (8) can be stated in terms of $\bar{F}_X(x)$,

$$Q_X(t,x) = \frac{\bar{F}_X(x+t)}{\bar{F}_X(t)}. \qquad (11)$$

- Denote by $\mathcal{Q}$ the class of all random variables for which $Q_X(t,x)$ is a *non-increasing* function of t for any $x \geq 0$, and by $\mathcal{Q}^*$ the class of all random variables for which $Q_X(t,x)$ is a *non-decreasing* function of t for any $x \geq 0$.
- The random variables belonging to $\mathcal{Q}$ satisfy inequality (9) and those belonging to $\mathcal{Q}^*$ satisfy inequality (10) for any $x \geq 0$.

Other stochastic dominance relations

- In case the random variable X has a density $f_X(x)$, then it can be determined whether it belongs to $\mathcal{Q}$ or $\mathcal{Q}^*$ by the behavior of the function

$$h_X(t) = \frac{f_X(t)}{\bar{F}_X(t)} \tag{12}$$

which is known as the *hazard rate function* or the *failure rate function*.

- If $h_X(t)$ is a non-increasing function, then $X \in \mathcal{Q}$. If it is a non-decreasing function, then $X \in \mathcal{Q}^*$.
- The only distribution which belongs to both classes is the exponential distribution. The hazard rate function of the exponential distribution is constant with respect to t.

Other stochastic dominance relations

Now we introduce a stochastic dominance order assuming that the random variables describe random profits.

- Denote by $\Lambda_X(t)$ the transform

$$\Lambda_X(t) = -\log(\bar{F}_X(t)). \tag{13}$$

- A positive random variable X is said to dominate another positive random variable Y with respect to the Λ transform, $X \succeq_\Lambda Y$, if the random variable $Z = \Lambda_Y(X)$ is such that $Z \in \mathcal{Q}$.

Other stochastic dominance relations

- The rationale behind the Λ transform is the following. Consider the special case $Y = X$. The r.v. $Z = \Lambda_Y(X)$ has exactly the exponential distribution because $\bar{F}_Y(X)$ is uniformly distributed.

  If Y has a heavier tail than X, then Z has a tail which increases no more slowly than the tail of the exponential distribution and, therefore, $Z \in \mathcal{Q}$.

  $\Rightarrow$ The stochastic order $\succeq_\Lambda$ emphasizes the tail behavior of X relative to Y.
- This stochastic order is interesting since it does not arise from a class of utility functions and it has application in finance describing choice under uncertainty. We illustrate this by showing a relationship with SSD.

Other stochastic dominance relations

- Suppose that $X \succeq_{\wedge} Y$. Then, Kalashnikov and Rachev (1990) show that the following condition holds

$$\int_x^\infty \bar{F}_X(x)dx \leq \int_x^\infty \bar{F}_Y(x)dx, \quad \forall x \geq 0. \tag{14}$$

- The converse statement is not true; that is, condition (14) does not ensure $X \succeq_{\wedge} Y$. Equation (14) can be directly connected with SSD. In fact, if (14) holds and we assume that the expected payoffs of X and Y are equal, then

$$\int_0^x F_X(x)dx \leq \int_0^x F_Y(x)dx, \quad \forall x \geq 0.$$

- This inequality means that X dominates Y with respect to RSD and, therefore, with respect to SSD. Thus, we have demonstrated that if $EX = EY$, then

$$X \succeq_{\wedge} Y \quad \Longrightarrow \quad X \succeq_{RSD} Y \quad \Longrightarrow \quad X \succeq_{SSD} Y. \tag{15}$$

Other stochastic dominance relations

- Suppose that the random variables describe losses (applied in operational risk management).
- We modify the stochastic order in the following way. A positive random variable X is said to dominate another positive random variable Y with respect to the Λ transform, $X \succeq_{\Lambda^*} Y$, if the random variable $Z = \Lambda_Y(X)$ is such that $Z \in \mathcal{Q}^*$. In this case, the tail of X is heavier than the tail of Y.

Other stochastic dominance relations

- If the random variables describe returns, then the left tail describes losses and the right tail describes profits.
- The random variable can be decomposed into two terms,

$$X = X_{+} - X_{-},$$

  where $X_{+} = \max(X, 0)$ stands for the profit and $X_{-} = \max(-X, 0)$ denotes the loss.
- By modifying the stochastic order, we can determine the tail of which of the two components influences the stochastic order.
- Consider two real valued random variables X and Y describing random returns. The order $\succeq_{\Lambda}$ compares the tails of the profits X_{+} and Y_{+}, and $\succeq_{\Lambda^*}$ compares the tails of the losses X_{-} and Y_{-}.

Other stochastic dominance relations

$\Rightarrow$ The stochastic orders $\succeq_\Lambda$ and $\succeq_{\Lambda^*}$ are constructed without considering first a particular class of investors but by imposing directly a condition on the tail of the random variable.

- There may or may not be a corresponding set of utility functions such that if $Eu(X) \geq Eu(Y)$ for all $u(x)$ in this class, then $X \succeq_\Lambda Y$, for example.
- We have demonstrated that the order $\succeq_\Lambda$ is consistent with SSD and is not implied by it.

$\Rightarrow$ The stochastic order can be defined without seeking first a class of investors which can generate it, but we can only search for a consistency relation with an existing stochastic order (Eq. (15)).

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Chapter 5.