Technical Appendix

Lecture 10: Performance measures

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Portfolio and Asset Liability Management

Summer Semester 2008

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The material is based on the text-book:

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Advanced Stochastic Models, Risk Assessment, and Portfolio Optimization: The Ideal Risk, Uncertainty, and Performance Measures

John Wiley, Finance, 2007

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Extensions of STARR

- We revisit the problem of finding the maximal STARR portfolio.
- We demonstrate that the Rachev ratio can be viewed as an extension of STARR.
- Furthermore, we show that a new performance measure extending STARR can be derived.

Extensions of STARR

- Consider the definition of STARR given in **(8)** in the lecture. In order to keep notation simpler, we denote the active portfolio return by $X = r_p - r_b$.
- STARR can be represented as

$$\begin{aligned} STARR_\epsilon(w) &= \frac{EX}{AVaR_\epsilon(X)} \\ &= \frac{-\epsilon AVaR_\epsilon(X) + \int_\epsilon^1 F_X^{-1}(p)dp}{AVaR_\epsilon(X)} \\ &= -\epsilon + (1-\epsilon)\frac{\frac{1}{1-\epsilon}\int_\epsilon^1 F_X^{-1}(p)dp}{AVaR_\epsilon(X)}. \end{aligned} \tag{1}$$

- The numerator in the ratio is the average active return provided that it is larger than the VaR at tail probability ϵ.

Extensions of STARR

- In fact, the fraction can be recognized as the Rachev ratio with $\epsilon_1 = 1 - \epsilon$ and $\epsilon_2 = \epsilon$,

$$RaR_{1-\epsilon,\epsilon}(w) = \frac{\frac{1}{1-\epsilon}\int_{\epsilon}^{1} F_X^{-1}(p)dp}{AVaR_{\epsilon}(X)}.$$

- As a consequence, the portfolios maximizing STARR also maximize the $RaR_{1-\epsilon,\epsilon}(w)$ as the former is a positive linear function of the latter which is the main conclusion in (1).
- Thus, from the standpoint of the ex-ante analysis, STARR and $RaR_{1-\epsilon,\epsilon}(w)$ can be regarded as equivalent performance measures.
- The more general Rachev ratio appears when $1 - \epsilon$ is replaced by an arbitrary probability ϵ_1.

Extensions of STARR

- The representation in (1) provides a way of obtaining another generalization of STARR which we call the robust STARR and abbreviate by RobS. It is defined as

$$RobS_{\delta,\epsilon}(w) = \frac{\frac{1}{\delta-\epsilon}\int_{\epsilon}^{\delta} F_X^{-1}(p)dp}{AVaR_{\epsilon}(X)} \tag{2}$$

  where $\delta \geq \epsilon$ is an upper tail probability.
- The numerator can be interpreted as the average active return between VaR at tail probability ϵ and the quantile at upper tail probability δ.
- Since the extreme quantiles are not included, the numerator can be viewed as a reward measure which is a robust alternative of the mathematical expectation.
- A reasonable choice for δ is, for example, $\delta = 0.95$.
- The optimal STARR portfolios appear from the optimal $RobS_{\delta,\epsilon}(w)$ portfolios when $\delta = 1$.

Extensions of STARR

- Taking advantage of the same approach as in the derivation of the representation in (1), it is possible to obtain that the optimal $RobS_{\delta,\epsilon}(w)$ portfolios also maximize the ratio,

$$RobS^{*}_{\delta,\epsilon}(w) = \frac{-AVaR_{\delta}(X)}{AVaR_{\epsilon}(X)}, \tag{3}$$

which means that (3) is equivalent to (2) as far as the ex-ante analysis is concerned.

- The formula in (3) turns out to be a more suitable objective function than (2). In effect, the optimal robust STARR problem is

$$\begin{array}{ll} \max\limits_{w} & \dfrac{-AVaR_{\delta}(r_p - r_b)}{AVaR_{\epsilon}(r_p - r_b)} \\ \text{subject to} & w'e = 1 \\ & w \geq 0. \end{array} \quad (4)$$

- The robust STARR is a quasi-concave performance measure which can be optimized through a linear programming problem.

Quasi-concave performance measures

- In this section, we consider the RR ratio optimization problem of the general form

$$\begin{array}{rl} \max\limits_{w} & \dfrac{\nu(w'X - r_b)}{\rho(w'X - r_b)} \\ \text{subject to} & w'e = 1 \\ & w \geq 0, \end{array} \qquad (5)$$

  where

X	is a random vector describing the return of portfolio assets
ν	is a reward measure
ρ	is a risk measure
r_b	is return of a benchmark portfolio

- Depending on the properties assumed for the reward measure and the risk measure, the optimization problem can be reduced to a simpler form.

Quasi-concave performance measures

We start with a few comments on the general properties of problem (5) which is also called a fractional program because the objective function is a ratio.

- *First*, in order for the objective function to be bounded, we have to assume that the denominator does not turn into zero for any feasible portfolio. For this reason, we assume that the risk of the active portfolio return is positive for all feasible portfolios. This assumption is crucial, if it does not hold, then the optimization problem does not have a solution.
- *Second*, without loss of generality, we assume that the reward measure is positive for all feasible portfolios. This may be regarded as a restrictive property but if it does not hold, then we can consider the optimization problem only on the subset of the feasible portfolios for which $\nu(w'X - r_b) \geq \epsilon > 0$. The portfolios with negative reward can be safely ignored because the optimal solution can never be among them on condition that there are feasible portfolios with positive reward.

Quasi-concave performance measures

- In summary, the basic assumptions for all feasible portfolios are the following,

$$\nu(w'X - r_b) > 0 \qquad \text{and} \qquad \rho(w'X - r_b) > 0. \tag{6}$$

- If they are satisfied, then we can consider either problem (5), in which we maximize the RR ratio, or problem

$$\begin{array}{rl} \min\limits_{w} & \dfrac{\rho(w'X - r_b)}{\nu(w'X - r_b)} \\ \text{subject to} & w'e = 1 \\ & w \geq 0, \end{array} \tag{7}$$

  in which we minimize the inverse ratio.

- Under the basic assumptions in (6), the portfolios solving problem (5) also solve problem (7).

Quasi-concave performance measures

- The portfolio yielding the optimal ratio in (5) can also be interpreted as a tangent portfolio, which is similar to the corresponding interpretation when the benchmark is a constant target.
- If r_b is a r.v., then the efficient frontier is generated by a R-R analysis with a reward measure ν and a risk measure ρ which are considered on the space of active portfolio returns.
- The efficient portfolios are obtained by solving optimization problem (5) but changing the objective function to

$$f(w) = \nu(w'X - r_b) - \lambda\rho(w'X - r_b)$$

where $\lambda \geq 0$ is the risk-aversion parameters.
- By varying λ and solving the optimization problem, we obtain the set of efficient portfolios.

Quasi-concave performance measures

- The portfolio yielding the maximal ratio appears as a tangent portfolio to the efficient frontier in the reward-risk plane.
- The benchmark return is taken into account by considering the risk and the reward of the active portfolio returns.
- In effect, the tangent line identifying the tangent portfolio passes through the origin. *Figure 1* shows a case in which the tangent portfolio is not unique.

Convex programming problem

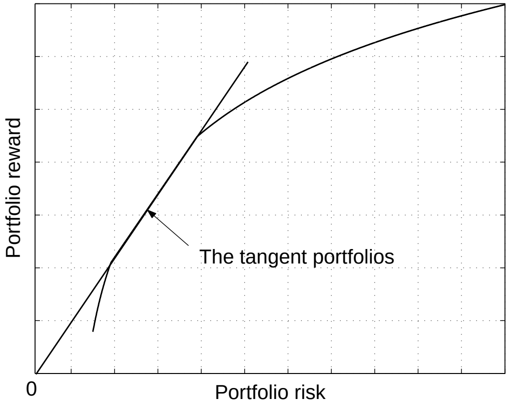

Figure 1. The efficient frontier may have a linear section which may result in non-unique tangent portfolios.

Quasi-concave fractional program

- If the reward functional is a concave function of portfolio weights and the risk measure is a convex function of portfolio weights, then the objective function of (5) is quasi-concave and the objective function of (7) is quasi-convex.
- If the reward measure satisfies the properties given in the appendix to *Lecture 8* and ρ is a coherent risk measure, then they are a concave and a convex function respectively.

Quasi-concave fractional program

- Quasi-concave and quasi-convex functions have nice optimality properties which are similar to the properties of the concave and convex functions, respectively.
- For example, if the objective function of (7) is quasi-convex then there exists a unique solution.
- The differences from the convex functions can be best illustrated if the function has a one-dimensional argument.
- A quasi-convex function has one global minimum and is composed of two monotonic sections. In contrast to convex functions, the monotonic sections may not be strictly monotonic; that is, the graph may have some "flat" sections which make the optimization a more involved affair.

Quasi-concave fractional program

- Generally, an optimization problem with a quasi-convex function can be decomposed into a sequence of convex feasibility problems.
- The sequence of feasibility problem can be obtained using the set

$$\mathcal{W}_t = \left\{ w : \left| \begin{array}{l} \rho(w^T r - r_b) - t\mu(w^T r - r_b) \leq 0 \\ w^T e = 1 \\ w \geq 0 \end{array} \right. \right\}$$

  where t is a fixed positive number.
- For a given t, the above set is convex and therefore we have a convex feasibility problem.

Quasi-concave fractional program

- A simple algorithm based on bisection can be devised so that the smallest t is found, t_{min}, for which the set is non-empty.
- If t_{min} is the solution of the feasibility problem, then $1/t_{min}$ is the value of the optimal ratio and the portfolios in the set

$$\mathcal{W}_{t_{min}} = \left\{ w : \left| \begin{array}{l} \rho(w^T r - r_b) - t_{min}\mu(w^T r - r_b) \leq 0 \\ w^T e = 1 \\ w \geq 0 \end{array} \right. \right\}$$

  are the optimal portfolios solving the fractional problem (7).
- The same set of portfolios also solve problem (5).

Convex programming problem

- Suppose that the reward measure is a concave function of portfolio weights and the risk measure is a convex function of portfolio weights.
- In addition, suppose that both functions are positively homogeneous,

$$\nu(hX) = h\nu(X) \quad \text{and} \quad \rho(hX) = h\rho(X)$$

  where $h > 0$.
- In this case, we can formulate two convex optimization problems equivalent to (5) and (7) respectively.
- The equivalent convex problems are obtained through the substitutions $t^{-1} = \rho(w'X - r_b)$ and $t^{-1} = \nu(w'X - r_b)$ for the former and the latter problem respectively and then setting $v = tw$.

Convex programming problem

- As a result, we obtain the problems

$$\begin{array}{rl} \max\limits_{v,t} & \nu(v'X - tr_b) \\ \text{subject to} & v'e = t \\ & \rho(v'X - tr_b) \leq 1 \\ & v \geq 0, t \geq 0 \end{array} \qquad (8)$$

and

$$\begin{array}{rl} \min\limits_{v,t} & \rho(v'X - tr_b) \\ \text{subject to} & v'e = t \\ & \nu(v'X - tr_b) \geq 1 \\ & v \geq 0, t \geq 0. \end{array} \qquad (9)$$

Convex programming problem

The equivalence with (5) and (7) respectively is the following.

- Suppose that the pair $(\bar{v}_1, \bar{t}_1)$ is an optimal solution to (8).
- Then, $\bar{w}_1 = \bar{v}_1/\bar{t}_1$ is a portfolio yielding the maximal ratio in (5). The quantity $1/\bar{t}_1$ is equal to the risk of the optimal portfolio.
- Furthermore, if we denote by ν_{max} the value of the objective function of (8) at the solution point $(\bar{v}_1, \bar{t}_1)$, then ν_{max} is equal to the value of the optimal ratio, i.e. the optimal value of the objective function of problem (5).
- As a consequence, $\nu_{max}/\bar{t}_1$ equals the reward of the optimal portfolio.

Convex programming problem

- In a similar way, if the pair $(\bar{v}_2, \bar{t}_2)$ is an optimal solution to (9), then $\bar{w}_2 = \bar{v}_2/\bar{t}_2$ is an optimal solution to (7) and, therefore, to (5).
- Denote by ρ_{min} the value of the objective function of (9) at the solution point $(\bar{v}_2, \bar{t}_2)$.
- Then, $1/\rho_{min}$ is equal to the value of the optimal ratio, i.e. the optimal value of the objective function of problem (5).
- In addition, $1/\bar{t}_2$ is the reward and $\rho_{min}/\bar{t}_2$ is the risk of the optimal portfolio.
- The portfolios $\bar{w}_1$ and $\bar{w}_2$ may not be the same because there may be many portfolios yielding the unique maximum of the fractional program (5).
- Geometrically, this case arises if the efficient frontier has a linear section and the tangent line passes through all points in the linear section. (See the illustration in *Figure 1*).

Convex programming problem

- As a sub-case in this section, suppose that both the risk and the reward measures satisfy the invariance property,

$$\nu(X + C) = \nu(X) + C \quad \text{and} \quad \rho(X + C) = \rho(X) - C. \tag{10}$$

  where C is an arbitrary constant.
- Under these assumptions and a few additional technical conditions given in the appendix to *Lecture 8*, we can associate an optimal RV ratio problem which is equivalent to (5) in the sense that both problems have coincident optimal solutions.

Convex programming problem

- Consider the following transformations of the objective function of (7),

$$
\begin{aligned}
\frac{\rho(w'X - r_b)}{\nu(w'X - r_b)} &= \frac{\rho(w'X - r_b - \nu(w'X - r_b)) - \nu(w'X - r_b)}{\nu(w'X - r_b)} \\
&= \frac{\rho(w'X - r_b - \nu(w'X - r_b))}{\nu(w'X - r_b)} - 1.
\end{aligned}
\tag{11}
$$

Convex programming problem

- In the appendix to *Lecture 8*, we demonstrated that the functional in the numerator

$$\rho(w'X - r_b - \nu(w'X - r_b))$$

can be a dispersion measure and, therefore, the ratio on the right hand-side is the inverse of a RV ratio.

- On the basis of equation (11) and the relationship between (7) and (5), we arrive at the conclusion that the optimal RV ratio problem

$$\begin{array}{ll} \max\limits_{w} & \dfrac{\nu(w'X - r_b)}{\rho(w'X - r_b - \nu(w'X - r_b))} \\ \text{subject to} & w'e = 1 \\ & w \geq 0, \end{array} \qquad (12)$$

has the same solution as the optimal RR ratio problem (5).

Convex programming problem

- A special example of an optimal ratio problem belonging to the category of convex programming problems is when the reward measure coincides with the mathematical expectation.
- In this case, the objective function of (5) and the reward constraint in (5) turn into linear functions. We only provide the corresponding version to (5) since the reward constraint can be an equality rather than an inequality,

$$\begin{array}{rl} \min\limits_{v,t} & \rho(v'X - tr_b) \\ \text{subject to} & v'e = t \\ & E(v'X) - tE(r_b) = 1 \\ & v \geq 0, t \geq 0. \end{array} \tag{13}$$

- In the case of a linear reward measure, the relationship between the optimal RV ratio problem (12) and the optimal RR ratio (5) explains the relationship between the RR ratios based on the expectations bounded coherent risk measures and the corresponding RV ratios based on deviation measures.

Convex programming problem

- Recall that the assumptions made for ρ are that it should be positive for all feasible portfolios, convex and positively homogeneous.
- Generally, these properties alone do not imply that ρ is a risk measure. (Any deviation measure satisfies them as well).
- As a consequence, the established relationship between (5) and (13) holds if there is a deviation measure in the denominator.
- Consider for instance the optimal Sharpe ratio problem **(30)** discussed in the lecture. The standard deviation in the denominator is a convex, positively homogeneous function of portfolio weights.
- The simpler convex programming problem, which is the analogue of (13), is problem **(31)** from the lecture.
- It can be further simplified to the quadratic programming problem **(32)** because of properties specific to the standard deviation.

Convex programming problem

- Another optimal portfolio problem falling into this category is the problem of maximizing the Sortino-Satchell ratio. The functional in the denominator is

$$\rho(w'X - r_b) = (E(s - (w'X - r_b))_+^q)^{1/q} \tag{14}$$

where $(x)_+^q = (\max(x, 0)^q)$, s is the minimum acceptable return level, and $q \geq 1$ is the order of the lower partial moment.

Convex programming problem

- Assuming that the portfolio weights sum up to 1, it turns out that this is a convex function of portfolio weights.
- In order to demonstrate this property, we consider (14) in the next more suitable form,

$$g(w) = (E(w'Z)_+^q)^{1/q} \tag{15}$$

where $Z = se - X + r_b e$ and $e = (1, \ldots, 1)$. In the demonstration, we refer to the celebrated Minkowski inequality.

Convex programming problem

- Consider a portfolio w_λ, which is a convex combination of two other portfolios; that is, $w_\lambda = \lambda w_1 + (1-\lambda)w_2$. Then,

$$\begin{aligned}
g(w_\lambda) &= (E((\lambda w_1 + (1-\lambda)w_2)'Z)_+^q)^{1/q} \\
&\leq (E(\lambda(w_1'Z)_+ + (1-\lambda)(w_2'Z)_+)^q)^{1/q} \\
&\leq (E(\lambda w_1'Z)_+^q)^{1/q} + (E((1-\lambda)w_2'Z)_+^q)^{1/q} \\
&= \lambda(E(w_1'Z)_+^q)^{1/q} + (1-\lambda)(E(w_2'Z)_+^q)^{1/q} \\
&= \lambda g(w_1) + (1-\lambda)g(w_2).
\end{aligned}$$

- The first inequality follows because of the convexity of the max function and in order to obtain the second inequality, we apply the Minkowski inequality.
- As a result, the function $g(w)$ is a convex function of portfolio weights.

Convex programming problem

- In addition to the convexity property, the function $g(w)$ is also positively homogeneous, $g(hw) = hg(w)$, $h > 0$.
- Therefore, the problem of maximizing the Sortino-Satchell ratio can be reduced to a problem of the type (13).
- The particular form of the simpler problem is

$$\begin{array}{rl} \min\limits_{v,t} & E(ts - v'X + tr_b)_+^q \\ \text{subject to} & v'e = t \\ & E(v'X) - tE(r_b) = 1 \\ & v \geq 0, t \geq 0, \end{array} \tag{16}$$

which is obtained after raising the objective function to the power $q \geq 1$. This transformation does not change the optimal solution points.

Convex programming problem

- If there are scenarios available for the assets returns and the benchmark return, then (16) can be further reduced to a more simple problem.
- In this case, the objective function is the estimator of the mathematical expectation and, therefore, it is a sum of maxima raised to the power q.
- The maxima are either positive or zero and can be replaced by additional variables following the method of linearizing a piece-wise linear convex function, which is used also in the linearization of AVaR described *Lecture 8*.

Convex programming problem

- In this reasoning, we consider the argument of the max function $ts - v'X + tr_b$ as a r.v. the scenarios of which are obtained from the scenarios of the assets returns and the benchmark return.
- As a result, we derive the optimization problem

$$\begin{array}{rl} \min\limits_{v,t,d} & \sum\limits_{i=1}^{k} d_i^q \\ \text{subject to} & tse - Hv + th_b \leq d \\ & v'e = t \\ & v'\mu - tEr_b = 1 \\ & v \geq 0, t \geq 0, d \geq 0. \end{array} \quad (17)$$

where $h_b = (r_b^1, \ldots, r_b^k)$ is a vector of the observed returns of the benchmark portfolio.

Convex programming problem

From the point of view of the optimal portfolio problem structure, there are two interesting cases.

- If $q = 1$, then (17) is a linear programming problem. This is not surprising because in this case the objective function in (16) is the expectation of the maxima function.
- If $q = 2$, then (17) is a quadratic programming problem. In this case, the objective function can be represented in matrix form as

$$\sum_{i=1}^{k} d_i^2 = d' I d,$$

where I stands for the identity matrix.

Reductions to linear programming problems

- Suppose that the reward measure is a concave function of portfolio weights and the risk measure is a convex function of portfolio weights, and that both functions are positively homogeneous.
- In addition to these properties, which were the basic assumptions in the previous section, suppose that both ν and ρ can be approximated by piece-wise linear functions.
- Then, the convex optimization problems (8) and (8) can be further simplified to linear programming problems.
- It is also often said that, in this case the convex problem allows for a linear relaxation.

Reductions to linear programming problems

- A problem belonging to this category is the optimal STARR problem. It arises when $\rho(X) = AVaR_\epsilon(X)$ and the reward measure is the mathematical expectation.
- On condition that there are scenarios for the assets returns and the benchmark return, AVaR can be approximated by a piece-wise linear function on the basis of which the convex optimization problem can be simplified to a linear programming problem.
- The linear programming problem can be directly applied to (13) by considering the argument of the risk measure $v'X - tr_b$ as a r.v. the scenarios of which are obtained from the scenarios of the assets returns X and the benchmark return r_b.

Reductions to linear programming problems

- Another problem in this category is the optimal robust STARR problem formulated in (4).
- The reward measure is the negative of AVaR at a certain upper tail probability and, therefore, it is a concave function of portfolio weights. The risk measure is AVaR. Both the reward measure and the risk measure can be linearized.
- In the case of the robust STARR, the analogue of the convex problem (9) is

$$\begin{array}{ll} \min\limits_{v,t} & AVaR_{\epsilon}(v'X - tr_b) \\ \text{subject to} & v'e = t \\ & AVaR_{\delta}(v'X - tr_b) \geq 1 \\ & v \geq 0, t \geq 0. \end{array} \quad (18)$$

Reductions to linear programming problems

- The linear relaxation of the convex optimization problem (18) is

$$
\begin{array}{rl}
\displaystyle\min_{(v,t,\theta_1,d,\theta_2,g)} & \theta_1 + \dfrac{1}{k\epsilon} d'e \\
\text{subject to} & -Hv - \theta_1 \leq d \\
& \theta_2 + \dfrac{1}{k\delta} g'e \leq 1 \\
& -Hv - \theta_2 \leq g \\
& v'e = t \\
& v \geq 0,\ t \geq 0,\ d \geq 0,\ g \geq 0 \\
& \theta_1 \in \mathbb{R},\ \theta_2 \in \mathbb{R},
\end{array}
\tag{19}
$$

where the auxiliary variables θ_1 and d are because of the linearization of the risk measure and the auxiliary variables θ_2 and g are because of the linearization of the reward measure.

The capital market line and quasi-concave ratios

- We considered the capital market line generated by mean-variance analysis with a risk-free asset added to the investment universe and the optimal Sharpe ratio problem.
- We demonstrated that the market portfolio, which is a key constituent of the efficient portfolios, yields the maximal Sharpe ratio with a constant benchmark return equal to the return on the risk-free asset.
- It turns out that this property is not valid only for the mean-variance analysis and the Sharpe ratio but also for the more general case of reward-risk analysis and the corresponding optimal quasi-concave ratio problem under certain technical conditions.
- The necessary general technical conditions are stated in the opening part of the section about quasi-concave ratio with the additional requirements that the reward measure and the risk measure are positively homogeneous and they satisfy the invariance property given in (10).

The capital market line and quasi-concave ratios

- Under these conditions, the optimal quasi-concave ratio problem (5) can be reduced to the convex problem

$$\begin{array}{rl} \min\limits_{v,t} & \rho(v'X) + tr_b \\ \text{subject to} & v'e = t \\ & \nu(v'X) - tr_b \geq 1 \\ & v \geq 0, t \geq 0. \end{array} \tag{20}$$

- We'll demonstrate that the two-fund separation theorem is valid for the efficient portfolios generated by reward-risk analysis with a risk-free asset added to the investment universe.
- Similar to the Sharpe ratio, the market portfolio appears as a solution to the optimal reward-to-risk ratio problem (5).

The capital market line and quasi-concave ratios

- The optimal portfolio problem behind reward-risk analysis with a risk-free asset is given by

$$\begin{array}{rl} \min\limits_{\omega,\omega_f} & \rho(\omega' X) - \omega_f r_f \\ \text{subject to} & \omega' e + \omega_f = 1 \\ & \nu(\omega' X) + \omega_f r_f \geq R_* \\ & \omega \geq 0, \omega_f \leq 1, \end{array} \quad (21)$$

where

ω	denotes the weights of the risky assets
ω_f	stands for the weight of the risk-free asset
r_f	denotes the return on the risk-free asset
R_*	denotes the bound on the expected portfolio return

- Negative values of ω_f are interpreted as borrowing at the risk-free rate with the borrowed funds invested in the risky assets. Also, we assume that the lower bound on the expected return is larger than the risk-free rate, $R_* > r_f$.

The capital market line and quasi-concave ratios

- We substitute the variable ω_f for $1 - s$ where s calculates the total weight of the risky assets in the portfolio.
- We derive the following optimization problem, equivalent to (21),

$$\begin{array}{rl} \min\limits_{\omega,s} & \rho(\omega' X) + s r_f - r_f \\ \text{subject to} & \omega' e = s \\ & \nu(\omega' X) - s r_f \geq R_* - r_f \\ & \omega \geq 0, s \geq 0. \end{array} \tag{22}$$

The capital market line and quasi-concave ratios

There are many similar features between the optimal ratio problem (20) and (22).

- Denote the optimal solution of (22) by $(\bar{\omega}, \bar{s})$. The optimal solution to (20) equals

$$\bar{v} = \frac{\bar{\omega}}{(R_* - r_f)} \quad \text{and} \quad \bar{t} = \frac{\bar{s}}{(R_* - r_f)}. \tag{23}$$

- Formula (23) holds because scaling the optimal solution $(\bar{\omega}, \bar{s})$ with the positive factor $1/(R_* - r_f)$ makes the resulting quantities feasible for problem (20).
- Furthermore, scaling the objective function of problem (22) by the same factor does not change the optimal solution point.

The capital market line and quasi-concave ratios

- Note that both $\bar{v}$ and $\bar{t}$, being an optimal solution to (20), do not depend on R_* because R_* is not a parameter in (20).
- The vector $\bar{v}$ and the scalar $\bar{t}$ can be regarded as characteristics of the efficient portfolios generated by (21).
- According to the analysis made for the generic optimal RR ratio problem (5), we obtain that the weights $\bar{w}$ of the portfolio yielding the maximal RR ratio are computed by

$$\bar{w} = \bar{v}/\bar{t} = \bar{\omega}/\bar{s} = \bar{\omega}/(1 - \bar{\omega}_f). \tag{24}$$

The capital market line and quasi-concave ratios

- As a consequence, the optimal RR ratio portfolio $\bar{w}$ is a fundamental ingredient in all portfolios in the efficient set generated by (21).
- The weights of the risky assets in the efficient portfolios are proportional to it and can be computed according to the formula $\bar{\omega} = \bar{w}(1 - \bar{\omega}_f)$.
- As a result, the optimal RR ratio portfolio represents the market portfolio and the returns of any reward-risk efficient portfolio with a risk-free asset can be expressed as

$$\bar{\omega}X + \bar{\omega}_f r_f = (1 - \bar{\omega}_f)\bar{w}X + \bar{\omega}_f r_f,$$

where $r_M = \bar{w}X$ stands for the return of the optimal RR ratio portfolio.

The capital market line and quasi-concave ratios

- The approach behind the derivation of the capital market line in the case of mean-variance analysis can be applied for the more general reward-risk analysis.
- We obtain that the equation for the capital market line is

$$\nu(r_p) = r_f + \left(\frac{\nu(r_M) - r_f}{\rho(r_M) + r_f}\right)(\rho(r_p) + r_f), \tag{25}$$

  where r_p denotes the return of the efficient portfolio.
- Equation (25) suggests that the capital market line coincides with the tangent line to the efficient frontier in the reward-shifted risk plane.

Non-quasi-concave performance measures

- Not all RR ratios and RV ratios belong to the class of the quasi-concave performance measures described in the previous section.
- Examples include the one-sided variability ratio defined in (**??**) and the Rachev ratio described in the chapter which are ratios of a convex reward measure of portfolio weights and a convex risk measure.
- Other examples include the generalized Rachev ratio, the Gini-type ratio, and the spectral-type ratio discussed in Rachev et al. (2007).

Non-quasi-concave performance measures

- Since these performance measures are not quasi-concave functions of portfolio weights, there may be multiple local extrema and, therefore, any numerical method based on convex programming will find the closest local maximum which may not be the global one.
- Nevertheless, for some of the non-quasi-concave performance measures, it could be possible to find a method yielding the global maximum.
- For example, in the case of the Rachev ratio, it is possible to find a mixed-integer programming problem finding the global maximum of the Rachev ratio.

Here we provide a definition of the generalized Rachev ratio as it includes several of the ratios as special examples.

- The generalized Rachev ratio is defined as

$$GRaR^{\delta,\gamma}_{\alpha,\beta}(w) = \frac{AVaR^{\delta}_{\alpha}(r_b - r_p)}{AVaR^{\gamma}_{\beta}(r_p - r_b)} \tag{26}$$

where α and β denote tail probabilities, and δ and γ are powers generalizing the AVaR concept,

$$AVaR^{\delta}_{\alpha}(X) = \left(\frac{1}{\alpha}\int_0^{\alpha}\left[\max(-F_X^{-1}(p), 0)\right]^{\delta} dp\right)^{1/\delta}.$$

in which $\delta \geq 1$ and X stands for the r.v. which in this case can be the active portfolio return $X = r_p - r_b$ or the negative of it $X = r_b - r_p$.

Non-quasi-concave performance measures

- If $\delta = 1$, then the quantity $AVaR_\alpha^\delta(X)$ coincides with AVaR,

$$AVaR_\alpha^1(X) = AVaR_\alpha(X).$$

  if $\alpha \leq F_X(0)$.

- As a consequence of this equality, the Rachev ratio appears as a special example of the generalized Rachev ratio,

$$GRaR_{\alpha,\beta}^{1,1}(w) = RaR_{\alpha,\beta}(w).$$

  when α and β are sufficiently small.

Non-quasi-concave performance measures

- Furthermore, choosing appropriately the tail probabilities, the generalized Rachev ratio generates a scaled one-sided variability ratio $\Phi_{r_b}^{p,q}(w)$.
- Suppose that $\alpha_1 = P(r_b - r_p \leq 0)$ and $\beta_1 = P(r_p - r_b \leq 0)$.
- Then, on condition that the active return is an absolutely continuous random variable,

$$GRaR_{\alpha_1,\beta_1}^{p,q}(w) = C.\Phi_{r_b}^{p,q}(w)$$

where $C = \beta_1^q / \alpha_1^p$ is a positive constant.

Probability metrics and performance measures

- Concerning the problem of evaluating the performance of a given portfolio, the ideas behind the theory of probability metrics can be applied in the construction of general families of performance measures.
- For example, consider the following general ratio,

$$GR^{\delta,\gamma}_{\alpha,\beta,M}(w) = \frac{AVaR^{\delta}_{\alpha,M}(r_b - r_p)}{AVaR^{\gamma}_{\beta,M}(r_p - r_b)} \quad (27)$$

where

$$AVaR^{\delta}_{\alpha,M}(X) = \left(\frac{1}{\alpha}\int_0^{\alpha}\left[\max(-F_X^{-1}(p), 0)\right]^{\delta} dM(p)\right)^{\min(1,1/\delta)} \quad (28)$$

in which $\delta > 0$ and all notation is the same as in formula (26) and the function $M(p)$ satisfies the properties of a cumulative distribution function (c.d.f.) of a r.v. defined in the unit interval.

There are a few interesting special cases of (28).

- If $M(p)$ is the c.d.f. of the uniform distribution in $[0, 1]$, then (28) coincides with the generalized Rachev ratio given in (26).
- As a next case, suppose that $M(p)$ is the c.d.f. of the constant α which is the tail probability in (28).
- Under this assumption, the integral equals the value of the integrand function at $p = \alpha$.
- As a result, we can obtain a performance measure represented by a scaled ratio of two VaRs,

$$GR^{1,1}_{\alpha,\beta,M}(w) = C\frac{VaR_{\alpha}(r_b - r_p)}{VaR_{\beta}(r_p - r_b)}$$

where $C = \beta/\alpha$ is a positive constant.

Furthermore, taking advantage of the underlying structure of the performance measure in (27), we can derive the next two limit cases.

- Suppose that $\delta \to \infty$ and $\gamma \to \infty$ and that $M(p)$ is a continuous function.
- Under these conditions and using the properties of the inverse c.d.f.,

$$GR_{\alpha,\beta,M}^{\infty,\infty}(w) = \frac{VaR_0(r_b - r_p)}{VaR_0(r_p - r_b)}$$

  where $VaR_0(X)$ denotes the smallest value that the random variable X can take.
- Thus, the performance measure $GR_{\alpha,\beta,M}^{\infty,\infty}(w)$ is in fact the ratio between the maximal outperformance of the benchmark and the maximal underperformance of the benchmark.
- This quantity does not depend on the selected tail probabilities and the form of the continuous c.d.f. $M(p)$.

- At the other limit, suppose that $\delta \to 0$ and $\gamma \to 0$. Then, using the properties of the inverse c.d.f., we derive the ratio

$$GR^{0,0}_{\alpha,\beta,M}(w) = \frac{\beta M(\alpha)}{\alpha M(\beta)}$$

  the properties of which are driven by the assumptions behind the c.d.f. $M(p)$.

- The general ratio defined in formula (27) can be regarded as an illustration of how the theory of probability metrics can be employed in order to obtain general classes of performance measures encompassing other performance measures as special cases.

Svetlozar T. Rachev, Stoyan Stoyanov, and Frank J. Fabozzi
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John Wiley, Finance, 2007.

Chapter 10.