Option pricing and hedging under a stochastic volatility Lévy process model

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Abstract

In this paper, we construct a new stochastic volatility model with a Lévy driving process and then apply the model to option pricing and hedging. The stochastic volatility in our model is defined by the continuous Markov chain. We use the Esscher transform to find the equivalent martingale measure. The option price using this model is obtained by the Fourier transform method. We obtain the closed-form solution for the hedge ratio by applying the local risk minimizing hedging.

Keywords: Option pricing; Hedging; Stochastic volatility; Continuous Markov chain; Regime-switching; Lévy process, Esscher transform

JEL Classifications: C6, G11, G12, G13

1 Introduction

The skewness and heavy-tail property observed for return distributions and the time-varying volatility of the return process are two major issues for modeling underlying return processes and option pricing. The skewness and heavy-tail property can be described by the non-Gaussian infinitely divisible distributions (see Rachev *et al.*, 2011). The time-varying volatility models for option pricing has been studied for the following three major classes of models: (1) continuous time and continuous market volatility, (2) discrete time and continuous market volatility, and (3) continuous time and finite market volatility.

The stochastic volatility model by Heston (1993) belongs to the first class since the model is a continuous-time model and the volatility in this model is defined on all positive real numbers. The stochastic volatility Lévy process model by Carr *et al.* (2003) is also included in the first class. Option pricing with the GARCH model as proposed by Duan (1995) is a discrete-time model, but volatility is defined on the all positive real numbers. Hence, the model is included in the second class. The model was subsequently enhanced by several researchers. For example, Menn and Rachev (2009), Kim *et al.* (2009), and Kim *et al.* (2010) replace the Gaussian innovation in Duan's model with the α-stable and the tempered stable innovation process. If market volatility is modeled by the finite states continuous Markov chain, then this model is referred to as a regime-switching model (see Buffington and Elliott (2002) and Elliott and Kopp (2010)) and belongs to the third class.

The main issue associated with option pricing is finding a risk-neutral measure; that is, finding an equivalent martingale measure (EMM) corresponding to the market probability measure[1] of the underlying stock return process. The Girsanov theorem has been successfully applied to find the EMM in time-varying volatility models having Brownian motion as a driving process such as the stochastic volatility model by Heston (1993) and the GARCH option pricing model by Duan (1995). However, the Girsanov theorem cannot be applied to general time-varying volatility models with Lévy driving processes. One classical method to find an EMM for non-Gaussian Lévy process models is the Esscher transform presented by Gerber and Shiu (1994); another reasonable method is finding the "minimal entropy martingale measure" presented by Fujiwara and Miyahara (2003). Kim and Lee (2007) discuss the EMM on a tempered stable process model with the generalized Girsanov theorem by Sato (1999). Finding an EMM with the Esscher transform for a regime-switching model with Brownian motion is addressed in Elliott *et al.* (2005).

In the Black-Scholes model (Black and Scholes, 1973), the market is assumed to be complete so that a perfect hedge can be attained by delta hedging. However, generally in an incomplete market, a perfect hedge is not attainable. Föllmer and Schweizer (1990) propose what they refer to as a local risk-minimizing method for constructing a portfolio containing options and underlying stocks that minimizes the variance based on the physical market measure. Boyarchenko and Levendorskiĭ (2002) discuss the local risk-minimizing hedging under the exponential Lévy stock price model and find an explicit form for the hedge ratio.

[1]The market probability measure is the probability measure for the physical price process of the underlying stock. Another name for the market probability measure is the physical measure. Typically the measure is estimated using the underlying stock's historical returns.

The purpose of this paper is threefold. First, we construct a stochastic volatility model with a Lévy driving process and then discuss the option pricing and hedging problems using this model. The stochastic volatility in our model is defined by the continuous Markov chain, the same approach as the regime-switching model. Although Jackson *et al.* (2007) also discuss regime-switching with Lévy driving process, their approach does not focus on stochastic volatility and therefore differs from our model. Second, we explain how to find the equivalent martingale measure (EMM) using the Esscher transform under the new stochastic volatility model and provide the European option pricing formula under this measure. Finally, we apply local risk-minimizing hedging for the stochastic volatility model with a Lévy driving process and present an explicit form for the hedge ratio.

The remainder of this paper is organized as follows. In Section 2, The Markov chain stochastic volatility Lévy process model is constructed and then Esscher transform are discussed. we provide the option pricing formula for the model in Section 3. Local risk minimizing hedging under the model is presented in Section 4. In Section 5, we summarize our principal findings.

2 Markov chain stochastic volatility Lévy process model and Esscher transform

Suppose $(L(t))_{t\geq 0}$ is a Lévy process on a complete probability space $(\Omega, \mathbb{P}, \mathcal{F})$. We assume that the states of the economy are modeled by a continuous-time hidden Markov chain process $(X(t))_{t\geq 0}$ generated by a generater matrix A on $(\Omega, \mathbb{P}, \mathcal{F})$ with a finite state space which is a finite set of unit vectors $\{e_1, e_2, \cdots, e_N\}$, where $e_n = (0, \cdots, 1, \cdots 0)' \in \mathbb{R}^N$. Then, we have the following semi-martingale representation theorem for $\{X(t)\}_{t\geq 0}$:

$$X(t) = X(0) + \int_0^t AX(s)ds + M(t), \tag{1}$$

where $(M(t))_{t\geq 0}$ is an $\mathbb{R}^N$-valued martingale increment process with respect to the filtration generated by $\{X(t)\}_{t\geq 0}$. We assume that $\{X(t)\}_{t\geq 0}$ and $\{L(t)\}_{t\geq 0}$ are independent and $E_{\mathbb{P}}[\exp(uL(t))] < \infty$ if $u \in I$ for some real interval I. We denote $\Phi(z) = \log E[\exp(zL(1))]$. The domain of $\Phi(z)$ is extended to a complex subset $\{z \in \mathbb{C} : \Re(z) = I\}$ by the analytic continuation in complex analysis (see Chruchill and Brown, 1990). Let $(\mathcal{F}_t^X)_{t\geq 0}$ and $(\mathcal{F}_t^L)_{t\geq 0}$ be the natural filtrations generated by $(X(t))_{t\geq 0}$ and $(L(t))_{t\geq 0}$, respectively. We define a filtration $(\mathcal{G}_t)_{t\geq 0}$ such that $\mathcal{G}_t$ is the σ-algebra generated by $\mathcal{F}_t^X$ and $\mathcal{F}_t^L$ for all $t \geq 0$.

The stock price process $(S(t))_{t\geq 0}$ is referred to as the *Markov chain stochastic volatility Lévy process model* if $S(t)$ is given by

$$S(t) = S(0)\exp\left(\int_0^t \langle \mu, X(s)\rangle ds - \int_0^t \Phi(\langle \sigma, X(s)\rangle)ds + \int_0^t \langle \sigma, X(s)\rangle dL(s)\right), \qquad (2)$$

where $\mu = (\mu_1, \mu_2, \cdots, \mu_N)' \in \mathbb{R}^N$ and $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N)' \in \mathbb{R}^N$ with $\sigma_n > 0$ and $\sigma_n \in I$ for each $n = 1, 2, \cdots, N$. The value $\mu(t) = \langle \mu, X(t)\rangle$ and $\sigma(t) = \langle \sigma, X(t)\rangle$ are referred to as the expected return and the market volatility at time t, respectively. The process $(S(t))_{t\geq 0}$ is $(\mathcal{G}_t)_{t\geq 0}$-adapted.

Lemma 1. *Let $0 \leq s \leq t$ and $y = (y_1, y_2, \cdots, y_N)' \in \mathbb{R}^N$. The conditional characteristic function of $\int_s^t \langle y, X(u)\rangle du$ under the σ-field $\mathcal{F}_s^X$ is equal to*

$$E\left[\exp\left(i\int_s^t \langle y, X(u)\rangle du\right)\Big|\mathcal{F}_s^X\right] = \langle \exp[(t-s)(A + \mathrm{diag}(iy))]X(s), \mathbf{1}\rangle \qquad (3)$$

where $(\mathcal{F}_t^X)_{t\geq 0}$ is the natural filtrations generated by $(X(t))_{t\geq 0}$ and $\mathbf{1} = (1, 1, \cdots, 1)' \in \mathbb{R}^N$.

Proof. Consider a process

$$Z(t) = \exp\left(i\int_s^t \langle y, X(u)\rangle du\right)X(t).$$

Then

$$dZ(t) = \exp\left(i\int_s^t \langle y, X(u)\rangle du\right) dX(t) + i\langle y, X(t)\rangle \exp\left(i\int_s^t \langle y, X(u)\rangle du\right) X(t)dt$$

By (1), we obtain

$$Z(t) - Z(s) = \int_s^t (A + i\text{diag}(y))Z(u)du + \int_s^t \exp\left(i\int_s^u \langle y, X(v)\rangle dv\right) dM_u.$$

Since $(M(t))_{t\geq 0}$ is martingale, $i\text{diag}(y) = \text{diag}(iy)$, and $Z(s) = X(s)$, we obtain the following equation by taking the conditional expectation

$$E[Z(t)|\mathcal{F}_s^X] = X(s) + \int_s^t (A + \text{diag}(iy))E[Z(u)|\mathcal{F}_s^X]du,$$

and hence

$$E[Z(t)|\mathcal{F}_s^X] = \exp\left((t-s)(A + \text{diag}(iy))\right) X(s).$$

Therefore we obtain (3) as follows

$$\begin{aligned} E\left[\exp\left(i\int_s^t \langle y, X(u)\rangle du\right) \Big| \mathcal{F}_s^X\right] &= E\left[\left\langle \exp\left(i\int_s^t \langle y, X(u)\rangle du\right) X(t), \mathbf{1}\right\rangle \Big| \mathcal{F}_s^X\right] \\ &= \langle \exp[(t-s)(A + \text{diag}(iy))]X(s), \mathbf{1}\rangle. \end{aligned}$$

□

Because the Markov process has a countable state space, the amount of time that the volatility state stays on each state e_n for $n = 1, 2, \cdots, N$ from 0 to time t is given as:

$$\tau_n^t = \int_0^t \langle e_n, X_u\rangle du, \tag{4}$$

where $\sum_{n=1}^N \tau_n^t = t$. By substituting $s = 0$ in equation (3) of Lemma 1, the characteristic function of the joint distribution of the random vector $(\tau_1^t, \tau_2^t, \cdots, \tau_N^t)$ is obtained by

$$E\left[\exp\left(i\sum_{k=1}^{N} y_k \tau_n^t\right)\right] = \langle \exp[t(A + \mathrm{diag}(iy))]X(0), \mathbf{1}\rangle, \tag{5}$$

where $y = (y_1, y_2, \cdots, y_N)'$, $\mathbf{1} = (1, 1, \cdots, 1)' \in \mathbb{R}^N$, and $X(0)$ is the initial state. For $0 \leq s \leq t$, we have

$$E\left[\exp\left(i\sum_{k=1}^{N} y_k (\tau_n^t - \tau_n^s)\right) \Big| \mathcal{F}_s^X\right] = \langle \exp[(t-s)(A + \mathrm{diag}(iy))]X(s), \mathbf{1}\rangle.$$

Lemma 2. *Let $0 \leq s \leq t$, $v = (v_1, v_2, \cdots, v_N)' \in \mathbb{R}^N$, and $w = (w_1, w_2, \cdots, w_N)' \in \mathbb{R}^N$ such that $\Phi(w_n) < \infty$ for all $n = 1, 2, \cdots, N$. Then*

$$\int_s^t \langle v, X(u)\rangle du = \sum_{n=1}^{N} v_n(\tau_n^t - \tau_n^s), \quad 0 \leq s \leq t \tag{6}$$

$$\int_s^t \langle w, X(u)\rangle dL(u) = \sum_{n=1}^{N} w_n L(\tau_n^t - \tau_n^s), \quad 0 \leq s \leq t. \tag{7}$$

and

$$E_{\mathbb{P}}\left[\exp\left(\sum_{n=1}^{N} v_n(\tau_n^t - \tau_n^s) + \sum_{n=1}^{N} w_n L(\tau_n^t - \tau_n^s)\right) \Big| \mathcal{G}_s\right]$$
$$= \langle \exp[(t-s)(A + \mathrm{diag}\gamma(x))]X(s), \mathbf{1}\rangle, \tag{8}$$

where $\gamma(x) = (v_1 + \Phi(w_1), v_2 + \Phi(w_2), \cdots, v_N + \Phi(w_N))'$.

Proof. Equations (6) and (7) are easy to prove. Equation (8) is proved using properties of the conditional expectation and independence between $L(t)$ and $\tau_n^t - \tau_n^s$ as follows:

$$\begin{aligned}
&E_{\mathbb{P}}\left[\exp\left(\sum_{n=1}^{N} v_n(\tau_n^t - \tau_n^s) + \sum_{n=1}^{N} w_n L(\tau_n^t - \tau_n^s)\right)\Big|\mathcal{G}_s\right] \\
&= E_{\mathbb{P}}\left[E_{\mathbb{P}}\left[\exp\left(\sum_{n=1}^{N} v_n(\tau_n^t - \tau_n^s) + \sum_{n=1}^{N} w_n L(\tau_n^t - \tau_n^s)\right)\Big|\tau_n^t - \tau_n^s\right]\Big|\mathcal{G}_s\right] \\
&= E_{\mathbb{P}}\left[\exp\left(\sum_{n=1}^{N}(v_n + \Phi(w_n))(\tau_n^t - \tau_n^s)\right)\Big|\mathcal{G}_s\right] \\
&= \langle \exp[(t-s)(A + \mathrm{diag}\gamma(x))]X(s), \mathbf{1}\rangle,
\end{aligned}$$

where $\gamma(x) = (v_1 + \Phi(w_1), v_2 + \Phi(w_2), \cdots, v_N + \Phi(w_N))'$. □

By (6) and (7), we have

$$S(t) = S(0)\exp\left(\sum_{n=1}^{N}\mu_n\tau_n^t - \sum_{n=1}^{N}\Phi(\sigma_n)\tau_n^t + \sum_{n=1}^{N}\sigma_n L(\tau_n^t)\right). \tag{9}$$

We consider a financial model consisting of two risky underlying assets, namely a bank account and a stock, which are tradable continuously. The risk-free rate of return, denoted by $r(t)$, at time $t \geq 0$ is given by $r(t) = \langle r, X(t)\rangle$, where $r = (r_1, r_2, \cdots, r_N)' \in \mathbb{R}^N$ with $r_n > 0$ for each $n = 1, 2, \cdots, N$ and $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^N$. The discounted stock price process $(\tilde{S}(t))_{t\geq 0}$ is defined by

$$\tilde{S}(t) = \exp\left(-\int_0^t \langle r, X(s)\rangle ds\right) S(t).$$

By (6) and (7), we have

$$\tilde{S}(t) = S(0)\exp\left(\sum_{n=1}^{N}(\mu_n - r_n)\tau_n^t - \sum_{n=1}^{N}\Phi(\sigma_n)\tau_n^t + \sum_{n=1}^{N}\sigma_n L(\tau_n^t)\right) \tag{10}$$

Proposition 1. *Let $Z(t) = \int_0^t \langle\sigma, X_s\rangle dL_s$, $t \geq 0$. If there exist $\theta = (\theta_1, \theta_2, \cdots, \theta_N) \in \mathbb{R}^N$*

which satisfies the condition

$$\begin{cases} \Phi(\theta_n\sigma_n) < \infty \\ \Phi((1+\theta_n)\sigma_n) < \infty \\ \mu_n - r_n = \Phi(\sigma_n) + \Phi(\theta_n\sigma_n) - \Phi((1+\theta_n)\sigma_n) \end{cases} \quad \textit{for all } n = 1, 2, \cdots, N, \tag{11}$$

then the measure $\mathbb{Q}_\theta$ *equivalent to the measure* $\mathbb{P}$ *with a Radon-Nikodym derivative*

$$\frac{d\mathbb{Q}_\theta}{d\mathbb{P}}\Big|_{\mathcal{G}_t} = \frac{\exp\left(\int_0^t \langle \theta, X(s)\rangle dZ(s)\right)}{E_{\mathbb{P}}\left[\exp\left(\int_0^t \langle \theta, X(s)\rangle dZ(s)\right) |\mathcal{F}_t^X\right]}, \ \textit{for } t \geq 0,$$

is an equivalent martingale measure (EMM).

Proof. Let $\xi_t = \frac{d\mathbb{Q}_\theta}{d\mathbb{P}}\big|_{\mathcal{G}_t}$. Then we have

$$\frac{\xi_T}{\xi_t} = \frac{\exp\left(\int_t^T \langle \theta, X(s)\rangle dZ(s)\right)}{\exp\left(\sum_{n=1}^N \Phi(\theta_n\sigma_n)(\tau_n^T - \tau_n^t)\right)} = \frac{\exp\left(\sum_{n=1}^N \theta_n\sigma_n L(\tau_n^T - \tau_n^t)\right)}{\exp\left(\sum_{n=1}^N \Phi(\theta_n\sigma_n)(\tau_n^T - \tau_n^t)\right)}$$

and $\frac{\xi_T}{\xi_t}$ is independent to $\mathcal{G}_t$. By (10), we have

$$\frac{\tilde{S}(T)}{\tilde{S}(t)} = \exp\left(\sum_{n=1}^N (\mu_n - r_n - \Phi(\sigma_n))(\tau_n^T - \tau_n^t) + \sum_{n=1}^N \sigma_n L(\tau_n^T - \tau_n^t)\right)$$

and $\frac{\tilde{S}(T)}{\tilde{S}(t)}$ is independent to $\mathcal{G}_t$. Hence we have

$$\begin{aligned} &E_{\mathbb{P}}\left[\tilde{S}(T)\frac{\xi_T}{\xi_t}|\mathcal{G}_t\right] = \tilde{S}(t)E_{\mathbb{P}}\left[\frac{\tilde{S}(T)}{\tilde{S}(t)} \cdot \frac{\xi_T}{\xi_t}|\mathcal{G}_t\right] \\ &= \tilde{S}(t)E_{\mathbb{P}}\left[\exp\left(\sum_{n=1}^N (\mu_n - r_n - \Phi(\sigma_n) - \Phi(\theta_n\sigma_n))(\tau_n^T - \tau_n^t) + \sum_{n=1}^N (1+\theta_n)\sigma_n L(\tau_n^T - \tau_n^t)\right)|\mathcal{G}_t\right] \\ &= \tilde{S}(t)E_{\mathbb{P}}\left[\exp\left(\sum_{n=1}^N (\mu_n - r_n - \Phi(\sigma_n) - \Phi(\theta_n\sigma_n) + \Phi((1+\theta_n)\sigma_n))(\tau_n^T - \tau_n^t)\right)|\mathcal{G}_t\right], \quad \text{by (8).} \end{aligned}$$

Therefore, if

$$\mu_n - r_n = \Phi(\sigma_n) + \Phi(\theta_n\sigma_n) - \Phi((1+\theta_n)\sigma_n), \quad \text{for } n = 1, 2, \cdots, N,$$

then

$$\tilde{S}(t) = E_{\mathbb{P}}\left[\tilde{S}(T)\frac{\xi_T}{\xi_t}|\mathcal{G}_t\right], \quad 0 \leq t \leq T, \tag{12}$$

and hence $\mathbb{Q}_\theta$ is an EMM corresponding to $\mathbb{P}$. □

The method to find an EMM using Proposition 1 is referred to as the Esscher transform under the Markov chain stochastic volatility Lévy process model. In the remainder of this paper, we denote $\xi_t = \frac{d\mathbb{Q}_\theta}{d\mathbb{P}}\big|_{\mathcal{G}_t}$. By the definition of $\frac{d\mathbb{Q}_\theta}{d\mathbb{P}}\big|_{\mathcal{G}_t}$ in Proposition 1, we obtain that

$$\xi_t = \exp\left(\sum_{n=1}^{N}(\theta_n\sigma_n L(\tau_n^t) - \Phi(\theta_n\sigma_n)\tau_n^t)\right). \tag{13}$$

3 Option pricing under the Markov chain stochastic volatility Lévy process model

In this section, we assume that $(S(t))_{t\leq 0}$ is given by the Markov chain stochastic volatility Lévy process model and $\mathbb{Q}_\theta$ is the EMM obtained by the Esscher transform. The expected return and and the market volatility for $(S(t))_{t\leq 0}$ is supposed to be $\mu = (\mu_1, \mu_2, \cdots, \mu_N)'$ and $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N)'$, respectively, the risk-free rate of return is supposed to be $r = (r_1, r_2, \cdots, r_N)'$, and we denote $Y(t) = \log(S(t)/S(0))$ in this section.

The conditional characteristic function of $Y(T) - Y(t)$ under measure $\mathbb{P}$ and the condition $\mathcal{G}_t$ is given by

$$\begin{aligned}\phi^{\mathbb{P}}_{Y(T)-Y(t)}(u|\mathcal{G}_t) &= E_{\mathbb{P}}[\exp(iu(Y(T)-Y(t)))|\mathcal{G}_t]\\ &= \langle\exp[(T-t)(A+\mathrm{diag}(y_{\mathbb{P}}(u)))]X(t), \mathbf{1}\rangle\end{aligned}$$

where

$$y_{\mathbb{P}}(z) = \begin{pmatrix} iu(\mu_1 - \Phi(\sigma_1)) + \Phi(iu\sigma_1) \\ iu(\mu_2 - \Phi(\sigma_2)) + \Phi(iu\sigma_2) \\ \vdots \\ iu(\mu_N - \Phi(\sigma_N)) + \Phi(iu\sigma_N) \end{pmatrix}. \tag{14}$$

Lemma 3. *Let $0 \leq t \leq T$. The conditional characteristic function of $Y(T) - Y(t)$ under measure $\mathbb{Q}_\theta$ and the condition $\mathcal{G}_t$ is given by*

$$\begin{aligned} \phi^{\mathbb{Q}_\theta}_{Y(T)-Y(t)}(u|\mathcal{G}_t) &= E_{\mathbb{Q}_\theta}[\exp(iu(Y(T) - Y(t)))|\mathcal{G}_t] \\ &= \xi_t \langle \exp[(T-t)(A + \mathrm{diag}(y_{\mathbb{Q}_\theta}(u)))]X(t), \mathbf{1}\rangle \end{aligned}$$

where ξ_t is given by (13) and

$$y_{\mathbb{Q}_\theta}(u) = \begin{pmatrix} iu(r_1 - \Phi((1+\theta_1)\sigma_1)) + \Phi((iu+\theta_1)\sigma_1) \\ iu(r_2 - \Phi((1+\theta_2)\sigma_2)) + \Phi((iu+\theta_2)\sigma_2) \\ \vdots \\ iu(r_N - \Phi((1+\theta_N)\sigma_N)) + \Phi((iu+\theta_N)\sigma_N) \end{pmatrix}. \tag{15}$$

Proof. We have

$$\begin{aligned} E_{\mathbb{Q}_\theta}[\exp(iu(Y(T) - Y(t)))|\mathcal{G}_t] &= \xi_t E_{\mathbb{P}}[\exp(iu(Y(T) - Y(t)))\xi_T/\xi_t|\mathcal{G}_t] \\ = \xi_t E_{\mathbb{P}}\Bigg[\exp\Bigg(iu \sum_{n=1}^{N} \big((\mu_n - \Phi(\sigma_n))(\tau_n^T - \tau_n^t) + \sigma_n L(\tau_n^T - \tau_n^t)\big)\Bigg) \\ \times \exp\Bigg(\sum_{n=1}^{N} \big(\theta_n \sigma_n L(\tau_n^T - \tau_n^t) - \Phi(\theta_n\sigma_n)(\tau_n^T - \tau_n^t)\big)\Bigg) \Bigg| \mathcal{G}_t \Bigg]. \end{aligned}$$

By the condition (11), we obtain

$$\begin{aligned}
&E_{\mathbb{Q}_\theta}[\exp(iu(Y(T)-Y(t)))|\mathcal{G}_t] \\
&=\xi_t E_{\mathbb{P}}\left[\exp\left(\sum_{n=1}^{N} iu(r_n-\Phi((1+\theta_n)\sigma_n))(\tau_n^T-\tau_n^t)+\sum_{n=1}^{N}(iu+\theta_n)\sigma_n L(\tau_n^T-\tau_n^t)\right)\Big|\mathcal{G}_t\right].
\end{aligned}$$

By (8), we complete the proof. □

For convenience, we define a function on $\mathbb{C}\times[0,\infty)$ where $\mathbb{C}$ is the complex field that

$$M_{r,\sigma,\theta}(z,t,T)=\langle\exp[(T-t)(A+\mathrm{diag}(y_{\mathbb{Q}_\theta}(z)))]X(t),\mathbf{1}\rangle,\quad z\in\mathbb{C}, 0\le t\le T,$$

where $y_{\mathbb{Q}_\theta}(z)$ is given by (15).

We obtain the option pricing formula under the Markov chain stochastic volatility Lévy process model by the Fourier transform method discussed in Carr and Madan (1999) and Lewis (2001) as follows.[2] Let $\Pi(S_T)$ be a payoff function of one European option with time to maturity T, $h(x)=\Pi(e^x)$ with $x=\log S(T)$, and $\hat{h}(\eta))=\int_{-\infty}^{\infty}e^{-i\eta x}h(x))dx$. Suppose $r=r_1=\cdots=r_N$ and $\hat{h}(\eta)$ is defined for all $\eta\in R_h$ where $R_h=\{z\in\mathbb{C}:\mathrm{Im}(z)\in I_h\}$ for some real interval I_h. The characteristic function $\phi_{Y(T)-Y(t)}^{\mathbb{Q}_\theta}(\eta)$ is defined for all $\eta\in R_\phi$ where $R_\phi=\{z\in\mathbb{C}:\mathrm{Im}(z)\in I_\phi\}$ for some real interval I_ϕ. Based on the no-arbitrage pricing framework, the European option price V with time to maturity T is given by

$$\begin{aligned}
V(t)&=E_{\mathbb{P}}\left[\exp(-r(T-t))\Pi(S(T))\frac{\xi_T}{\xi_t}|\mathcal{G}_t\right]\\
&=\frac{1}{\xi_t}E_{\mathbb{Q}_\theta}\left[e^{-r(T-t)}\Pi(e^{\log S(t)+Y(T)-Y(t)})|\mathcal{G}_t\right]\\
&=\frac{e^{-r(T-t)}}{\xi_t}E_{\mathbb{Q}_\theta}\left[h(\log S(t)+Y(T)-Y(t))|\mathcal{G}_t\right].
\end{aligned}$$

[2] This method is discussed in Liu *et al.* (2006) under the regime-switching model with Brownian motion.

Since the probability density function $f^{\mathbb{Q}_\theta}_{Y(T)-Y(t)}(x|\mathcal{G}_t)$ of $Y(T)-Y(t)$ under measure $\mathbb{Q}_\theta$ and the condition $\mathcal{G}_t$ can be obtained by

$$f^{\mathbb{Q}_\theta}_{Y(T)-Y(t)}(x|\mathcal{G}_t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-i(u+i\rho)x)\phi^{\mathbb{Q}_\theta}_{Y(T)-Y(t)}(u+i\rho|\mathcal{G}_t)du$$

using the complex inversion formula, we compute

$$\begin{aligned}
&E_{\mathbb{Q}_\theta}\left[h(Y(T)-Y(t))|\mathcal{G}_t\right] \\
&= \int_{-\infty}^{\infty} h(\log S(t)+x)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(u+i\rho)x}\phi^{\mathbb{Q}_\theta}_{Y(T)-Y(t)}(u+i\rho|\mathcal{G}_t)du\,dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(u+i\rho)(x-\log S(t))}h(x)dx\,\phi^{\mathbb{Q}_\theta}_{Y(T)-Y(t)}(u+i\rho|\mathcal{G}_t)du,
\end{aligned}$$

and hence

$$V(t) = \frac{e^{-r(T-t)}}{2\pi\xi_t}\int_{-\infty}^{\infty} e^{i(u+i\rho)\log S(t)}\hat{h}(u+i\rho)\phi^{\mathbb{Q}_\theta}_{Y(T)-Y(t)}(u+i\rho|\mathcal{G}_t)du, \tag{16}$$

where $\rho \in I_h \cap I_\phi$. By the Lemma 3, we obtain the European option pricing formula under the Markov chain stochastic volatility Lévy process model:

$$V(t) = \frac{e^{-r(T-t)}}{2\pi}\int_{-\infty}^{\infty} e^{i(u+i\rho)\log S(t)}\hat{h}(u+i\rho)M_{r,\sigma,\theta}(u+i\rho,t,T)du. \tag{17}$$

In particular, the payoff function of a European call option with time to maturity T and strike price K is given by $\Pi(S(T)) = (S(T)-K)^+$ and hence $h(x) = (e^x - K)^+$, where $(\cdot)^+ = \max(\cdot, 0)$. Therefore, we have

$$\hat{h}(\eta) = -\frac{Ke^{-i\eta\log K}}{\eta(i+\eta)}, \tag{18}$$

where $\eta \in \{u+i\rho|\rho < -1, u \in \mathbb{R}\}$. By substituting (18) into (17), a European call option pricing formula at time t is equal to

$$V(t) = \frac{e^{-r(T-t)}K^{1+\rho}}{2\pi S(t)^{\rho}} \int_{-\infty}^{\infty} \frac{e^{iu\log(S(t)/K)}}{(\rho - iu)(1+\rho - iu)} M_{r,\sigma,\theta}(u + i\rho, t, T)du, \tag{19}$$

where ρ is real number such that $\rho < -1$ and $y_n(u+i\rho) < \infty$ for all $n \in \{1, 2, \cdots, N\}$ and $u \in \mathbb{R}$. Moreover, by the property that $\hat{h}(-\eta; S(t)) = \overline{\hat{h}(\eta; S(t))}$ for the payoff function of a European call option, (19) becomes

$$V(t) = \frac{e^{-r(T-t)}K^{1+\rho}}{\pi S(t)^{\rho}} \mathrm{Re} \int_{0}^{\infty} \frac{e^{iu\log(S(t)/K)}}{(\rho - iu)(1+\rho - iu)} M_{r,\sigma,\theta}(u + i\rho, t, T)du. \tag{20}$$

We can compute a European put option's price in the same way as for a European call option. A European put option's price can be obtained by the same formula as (20), but the condition of ρ is different; that is, ρ is a strictly positive real number such that $y_n(u + i\rho) < \infty$ for all $n \in \{1, 2, \cdots, N\}$ and $u \in \mathbb{R}$.

4 Local risk-minimizing hedge ratio under the Markov chain stochastic volatility Lévy process model

The Markov chain stochastic volatility Lévy process model implies incomplete market. In an incomplete market, the perfect hedge for a general claim is no longer possible. In this section, we present the locally risk-minimizing hedge ratio which is discussed in Boyarchenko and Levendorskiĭ (2002) and deduce the explicit form of the hedge ratio in the Markov chain stochastic volatility Lévy process model.

Consider a European option with maturity $T > 0$ and let $V(t)$ be the European option's price at t. Let $W_{t+\Delta t}(\varphi) = V(t + \Delta t) - \varphi S_{t+\Delta t}$ at time $t < T$. Let $\varphi_{\Delta t}(t)$ be the share of the underlying stock which minimizes variance of $W_{t+\Delta t}(\varphi)$. That is

$$\varphi_{\Delta t}(t) = \arg\min_{\varphi} E_{\mathbf{P}}[(W_{t+\Delta t}(\varphi) - E_{\mathbf{P}}[W_{t+\Delta t}(\varphi)|\mathcal{G}_t])^2 \, |\mathcal{G}_t]. \tag{21}$$

The *locally risk-minimizing hedge ratio* φ_t at time t is defined as

$$\varphi(t) = \lim_{\Delta t \downarrow 0} \varphi_{\Delta t}(t).$$

In this section, we assume that $X(t + \Delta t) = X(t)$ for small Δt.

Proposition 2. *Let $\theta = (\theta_1, \theta_2, \cdots, \theta_N)' \in \mathbb{R}^N$ which satisfies the condition (11) and $\Phi(2\sigma_n)$ for all $n \in \{1, 2, \cdots, N\}$ and let $\rho \in \mathbb{R}$ satisfy $\Phi((1-\rho)\sigma_n) < \infty$ and $\Phi(-\rho\sigma_n) < \infty$ for all $n \in \{1, 2, \cdots, N\}$. The locally risk-minimizing hedging ratio is equal to*

$$\begin{aligned}\varphi(t) = &\frac{e^{-r(T-t)}}{2\pi S(t)\langle(\mathrm{diag}(d(\sigma)) - A)X(t), \mathbf{1}\rangle} \\ &\times \int_{-\infty}^{\infty} e^{i(u+i\rho)Y(t)} \hat{h}(u + i\rho) M_{r,\sigma,\theta}(u + i\rho, t, T) \langle(\mathrm{diag}(y_H(u + i\rho)) - A)X(t), \mathbf{1}\rangle du,\end{aligned} \tag{22}$$

where

$$d(\sigma) = \begin{pmatrix} \Phi(2\sigma_1) - 2\Phi(\sigma_1) \\ \Phi(2\sigma_2) - 2\Phi(\sigma_2) \\ \vdots \\ \Phi(2\sigma_N) - 2\Phi(\sigma_N) \end{pmatrix} \text{ and } y_H(z) = \begin{pmatrix} \Phi((iz + 1)\sigma_1) - \Phi(iz\sigma_1) \\ \Phi((iz + 1)\sigma_2) - \Phi(iz\sigma_2) \\ \vdots \\ \Phi((iz + 1)\sigma_N) - \Phi(iz\sigma_N) \end{pmatrix}$$

for $z \in \mathbb{C}$.

Proof. Since we have

$$
\begin{aligned}
&E_{\mathbf{P}}[(W_{t+\Delta t}(\varphi) - E_{\mathbf{P}}[W_{t+\Delta t}(\varphi)|\mathcal{G}_t])^2 \,|\mathcal{G}_t] \\
&= \varphi_{\Delta t}(t)^2 \left(E_{\mathbb{P}}[S(t+\Delta t)^2|\mathcal{G}_t] - (E_{\mathbb{P}}[S(t+\Delta t)|\mathcal{G}_t])^2 \right) \\
&\quad - 2\varphi_{\Delta t}(t)(E_{\mathbb{P}}[S(t+\Delta t)V(t+\Delta t)|\mathcal{G}_t] - E_{\mathbb{P}}[S(t+\Delta t)|\mathcal{G}_t]E_{\mathbb{P}}[V(t+\Delta t)|\mathcal{G}_t]) \\
&\quad + E_{\mathbb{P}}[(V(t+\Delta t)) - E_{\mathbb{P}}[V(t+\Delta t)|\mathcal{G}_t])^2|\mathcal{G}_t],
\end{aligned}
$$

it minimizes at

$$\varphi_{\Delta t}(t) = \frac{F_t(\Delta t)}{G_t(\Delta t)}, \tag{23}$$

where

$$F_t(\Delta t) = E_{\mathbb{P}}[S(t+\Delta t)V(t+\Delta t)|\mathcal{G}_t] - E_{\mathbb{P}}[S(t+\Delta t)|\mathcal{G}_t]E_{\mathbb{P}}[V(t+\Delta t)|\mathcal{G}_t]$$

and

$$G_t(\Delta t) = E_{\mathbb{P}}[S(t+\Delta t)^2|\mathcal{G}_t] - (E_{\mathbb{P}}[S(t+\Delta t)|\mathcal{G}_t])^2.$$

We now observe that

$$
\begin{aligned}
E_{\mathbb{P}}[S(t+\Delta t)^2|\mathcal{G}_t] &= S(t)^2 E_{\mathbb{P}}[e^{2(Y(t+\Delta t)-Y(t))}|\mathcal{G}_t] \\
&= S(t)^2 \langle \exp[\Delta t(\mathrm{diag}(\gamma) + A)] X(t), \mathbf{1} \rangle \\
&= S(t)^2 (\langle X(t), \mathbf{1} \rangle + \Delta t \langle (\mathrm{diag}(\gamma) + A) X(t), \mathbf{1} \rangle) + \mathcal{O}(\Delta t^2)
\end{aligned}
$$

with

$$\gamma = \begin{pmatrix} 2\mu_1 - 2\Phi(\sigma_1) + \Phi(2\sigma_1) \\ 2\mu_2 - 2\Phi(\sigma_2) + \Phi(2\sigma_2) \\ \vdots \\ 2\mu_N - 2\Phi(\sigma_N) + \Phi(2\sigma_N) \end{pmatrix}$$

and

$$
\begin{aligned}
(E_{\mathbb{P}}[S(t+\Delta t)|\mathcal{G}_t])^2 &= S(t)^2\left(E_{\mathbb{P}}[e^{Y(t+\Delta t)-Y(t)}|\mathcal{G}_t]\right)^2 \\
&= S(t)^2\left(\langle \exp[\Delta t(\operatorname{diag}(\mu)+A)]X(t),\mathbf{1}\rangle\right)^2 \\
&= S(t)^2(\langle X(t),\mathbf{1}\rangle + 2\Delta t\langle(\operatorname{diag}(\mu)+A)X(t),\mathbf{1}\rangle) + \mathcal{O}(\Delta t^2).
\end{aligned}
$$

Hence we have

$$
G_t(\Delta t) = S(t)^2\Delta t\langle(\operatorname{diag}(d(\sigma))-A)X(t),\mathbf{1}\rangle + \mathcal{O}(\Delta t^2), \tag{24}
$$

where

$$
d(\sigma) = \begin{pmatrix} \Phi(2\sigma_1)-2\Phi(\sigma_1) \\ \Phi(2\sigma_2)-2\Phi(\sigma_2) \\ \vdots \\ \Phi(2\sigma_N)-2\Phi(\sigma_N) \end{pmatrix}.
$$

By equation (16), we have

$$
\begin{aligned}
&E_{\mathbb{P}}[S(t+\Delta t)V(t+\Delta t)\ |\mathcal{G}_t] \\
&= \frac{S(t)e^{-r(T-t)}}{2\pi}\int_{-\infty}^{\infty} e^{i(u+i\rho)Y(t)}\hat{h}(u+i\rho) \\
&\qquad\qquad E_{\mathbb{P}}\left[e^{i(u+i(\rho-1))\Delta Y(t)+r\Delta t}M_{r,\sigma,\theta}(u+i\rho,t+\Delta t,T)\ |\mathcal{G}_t\right]du,
\end{aligned}
$$

where $\Delta Y(t) = Y(t+\Delta t)-Y(t)$. Since we can prove that

$$
\begin{aligned}
&M_{r,\sigma,\theta}(u+i\rho,t+\Delta t,T) \\
&= M_{r,\sigma,\theta}(u+i\rho,t,T)(1-\Delta t\langle(\operatorname{diag}(y_{\mathbb{Q}_\theta}(u+i\rho))+A)X(t),\mathbf{1}\rangle) + \mathcal{O}(\Delta t^2).
\end{aligned}
$$

under the assumption $X(t+\Delta t) = X(t)$ for small Δt where $y_{\mathbb{Q}_\theta}$ is given by the equation

(15), we obtain that

$$
\begin{aligned}
&E_{\mathbb{P}}[S(t{+}\Delta t)V(t+\Delta t)\ |\mathcal{G}_t] \\
&\quad = \frac{S(t)e^{-r(T-t)}}{2\pi}\int_{-\infty}^{\infty} e^{i(u+i\rho)Y(t)}\hat{h}(u+i\rho)M_{r,\sigma,\theta}(u+i\rho,t,T) \\
&\qquad \times \big(1 + r\Delta t - \Delta t\langle(\mathrm{diag}(y_{\mathbb{Q}_\theta}(u+i\rho)) + A)X(t), \mathbf{1}\rangle \\
&\qquad + \Delta t\langle(\mathrm{diag}(y_{\mathbb{P}}(u+i(\rho-1)) + A)X(t), \mathbf{1}\rangle\big)du \\
&\quad + \mathcal{O}(\Delta t^2),
\end{aligned}
$$

where $y_{\mathbb{P}}$ is given in the equation (14). By the same argument, we obtain the second part that

$$
\begin{aligned}
&E_{\mathbb{P}}[S(t+\Delta t)\ |\mathcal{G}_t]\, E_{\mathbb{P}}[V(t+\Delta t)\ |\mathcal{G}_t] \\
&= \frac{S(t)e^{-r(T-t)}}{2\pi}\int_{-\infty}^{\infty} e^{i(u+i\rho)Y(t)}\hat{h}(u+i\rho)M_{r,\sigma,\theta}(u+i\rho,t,T) \\
&\quad \times (1 + r\Delta t + \Delta t\langle(\mathrm{diag}(\mu) + A)X(t), \mathbf{1}\rangle - \Delta t\langle(\mathrm{diag}(y_{\mathbb{Q}_\theta}(u+i\rho)) + A)X(t), \mathbf{1}\rangle \\
&\quad + \Delta t\langle(\mathrm{diag}(y_{\mathbb{P}}(u+i\rho)) + A)X(t), \mathbf{1}\rangle)du + \mathcal{O}(\Delta t^2)
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
F_t(\Delta t) = \frac{S(t)e^{-r(T-t)}}{2\pi} &\Delta t\int_{-\infty}^{\infty} e^{i(u+i\rho)Y(t)}\hat{h}(u+i\rho)M_{r,\sigma,\theta}(u+i\rho,t,T) \\
&\times \langle(\mathrm{diag}(y_{\mathbb{P}}(u+i\rho-i) - y_{\mathbb{P}}(u+i\rho) - \mu) - A)X(t), \mathbf{1}\rangle du \\
&+ \mathcal{O}(\Delta t^2). \qquad (25)
\end{aligned}
$$

By the definition of $y_{\mathbb{P}}$, we have

$$y_{\mathbb{P}}(z-i) - y_{\mathbb{P}}(z) - \mu = \begin{pmatrix} \Phi((iz+1)\sigma_1) - \Phi(iz\sigma_1) \\ \Phi((iz+1)\sigma_2) - \Phi(iz\sigma_2) \\ \vdots \\ \Phi((iz+1)\sigma_N) - \Phi(iz\sigma_N) \end{pmatrix}$$

for $z \in \mathbb{C}$.

Substituting (24) and (25) into (23), we obtain

$$\varphi_{\Delta t}(t) = \frac{e^{-r(T-t)} \int_{-\infty}^{\infty} e^{i(u+i\rho)Y(t)} \hat{h}(u+i\rho) M_{r,\sigma,\theta}(u+i\rho, t, T) \Psi(u+i\rho, X(t)) du + \mathcal{O}(\Delta t)}{2\pi S(t) \langle (\mathrm{diag}(d(\sigma)) - A) X(t), \mathbf{1} \rangle + \mathcal{O}(\Delta t)},$$

where $\Psi(z,x) = \langle (\mathrm{diag}(y_{\mathrm{H}}(z)) - A)x, \mathbf{1} \rangle$. Taking $\Delta t \to 0$, we obtain (22). □

5 Conclusion

In this paper we construct the Markov chain stochastic volatility Lévy process model. The model is stochastic volatility model with a Lévy driving process and a continuous Markov chain volatility process. The EMM of the model is defined by the Esscher transform and option price formula is obtained by the Fourier transform method. The locally risk-minimizing hedging ratio is also discussed and finally the closed form solution of the hedge ratio is presented.

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