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Practical Portfolio Selection Problems Consistent With A Given Preference Ordering

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In this paper, we examine three portfolio-type problems where investors rank their choices considering each of the following: (1) risk, (2) uncertainty, and (3) the distance from a benchmark. For each problem, we analyze possible orderings for the choices and we propose several admissible portfolio optimization problems. Thus, we discuss the properties of several ─ risk measures, uncertainty measures and tracking error measures ─ and their consistency with investor choices. Furthermore, we propose several linearizable allocation problems consistent with a given ordering and demonstrate how many portfolio selection problems proposed in literature can be solved.

Key words: Probability metrics, tracking error measures, stochastic orderings, coherent measures, linearizable optimization problems, behavioral finance ordering.

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1. Introduction

The purpose of this paper is twofold. First, we show how to use the connection between ordering theory and the theory of probability functionals in portfolio selection problems. Second, we discuss the computational complexity of selection problems consistent with the preferences of investors. With these purposes in mind, we review several single-period portfolio problems proposed in the literature, emphasizing those which are computational simple (portfolio problems that can be reduced at least to convex programming problems) for different categories of tracking error measures, uncertainty measures, and risk measures.

Portfolio selection problems can be distinguished and classified based on motivations and intentions of investors. So, we generally refer to reward-risk problems when investors balance the advantage and disadvantage of a choice in a reward-risk space optimizing a probability functional that considers both measures. Instead, we refer to target-based approaches (or tracking-error type portfolio problems) when investors want to optimize a distance with respect to some financial benchmarks. Moreover, in other cases it could be important to optimize a measure of randomness of a given portfolio in order to maximize the returns of the portfolio choices. In all these portfolio selection approaches, we should rank different investor preferences and it is for this reason ordering theory provides some intuitive rules that are consistent with utility theory under uncertainty conditions. In particular, recent research classifies portfolio selection problems with respect to investors' preference orderings.[1]

The first macro-classification is between risk ordering/measure and uncertainty ordering/measure. Typically, we refer to risk orderings as the stochastic orderings between random variables that are implied by the monotony order.[2] Uncertainty orderings characterize the different degrees of uncertainty and dispersion. Since the uncertainty and the dispersion of a variate X is referred to the randomness of X we should expect its presence in the same (or proportional) quantity in its opposite $-X$**.** Thus**,** we say that X exhibits higher uncertainty than another variate Y when X dominates Y for a given risk ordering and also $-X$ dominates $-Y$ with respect to the same or another risk ordering. Roughly speaking, we define risk (uncertainty) measures as all probability functionals consistent with a risk (uncertainty) ordering; that is, if a random variable X is preferred to Y with respect to a given risk (uncertainty) ordering, this implies that the risk (uncertainty) measure of X should be lower than the measure of Y. Vice versa, a reward measure is a functional v isotonic with a risk ordering, i.e., if $X>Y$, implies that $v(X) \geq v(Y)$. However, we do not know *a priori* all possible risk/uncertainty orderings.

[1] See, among others, Rachev *et al.* (2008) and Ortobelli *et al.* (2008).
[2] X is preferred to Y with respect to the monotony order iff $X>Y$.

Thus, we cannot say *a priori* that a measure is consistent with a risk or an uncertainty ordering. Consequently, in recent years, several papers have proffered alternative definitions of risk/uncertainty measures based on some properties that serve to identify the risk and the uncertainty (see, among others, Artzner *et al.* (1999)). For most of these measures, it is also possible to prove their consistency with risk/uncertainty orderings.

As demonstrated by many studies in behavioral finance,[3] not all investors are non-satiable and risk-averse and, for this reason, it is important to classify the optimal choices for any admissible ordering of preferences. Once we know the kind of ordering be used for the portfolio problem, we have to identify a probability functional (referred to as a FORS probability functional) that characterizes an ordering among the admissible choices and that is consistent with investors preferences. Therefore, as suggested by Ortobelli *et al.* (2008), we get choices that are efficient in the sense of the preference ordering when we opportunely optimize the probability functional associated with that ordering. However, several new questions arise from this analysis. The main contribution of this paper is that it answers the following two questions:

- What is the "right" ordering of preferences that should be used?
- Can we describe some practical optimization problems for different orderings of preferences that (at least in some cases) can be used even for large-scale portfolios?

To answer these questions, we propose and discuss several new portfolio selection models. In particular, we investigate on the relationships between FORS probability functionals and the theory of probability metrics, and we demonstrate that many recent results on risk measures and portfolio selection can be seen as particular cases of the results presented in this paper. An answer to the question about the "right" ordering of preferences should be obviously partial, and depending on the particular investor's problem. So, we review several orderings utilized in portfolio theory and introduce new potential orderings of preferences. In the paper, we manly discuss the solutions to the following three specific portfolio problems associated with many possible orderings of preference.

[3] See, among others, Levy and Levy (2002) and the references therein.

1. Suppose an investor wants to outperform a benchmark taking into account the distributional characteristics of portfolio returns. In this case, we should use a risk ordering among portfolios taking into account the performance with respect to the benchmark. Moreover, a good asset manager should choose the opportune ordering for this active tracking error strategy.
2. Suppose an investor wants to maximize his/her profit by purchasing a portfolio of put and call options on some asset class indexes. From option pricing theory, we know that if we maximize the concentration of the underlying log-return indexes, we implicitly optimize the investor's opportunities to exercise the options. So, in this case, the asset manager should use an uncertainty ordering on the underlying log-return indexes taking into account the performance of the portfolio return on the derivatives.
3. Alternatively, suppose an investor seeks to track a benchmark as closely as possible (i.e., follows an indexing strategy). In this case, we should order the distance with respect to the benchmark. Thus, a good asset manager should minimize an opportune probability functional to track the benchmark with a portfolio of returns.

As expected, the three portfolio problems above suggest to use different orderings among random variables. In this paper, we try to identify different possible answers to these three problems, distinguishing those practical optimization problems for different orderings of preferences that in some cases could be used for large-scale portfolios.

We have organized the paper as follows. In Section 2, we review the concept of FORS orderings and derive practical reward-risk models based on measures consistent with the most known orders. In Section 3, we begin by describing practical portfolio selection problems consistent with uncertainty orderings. Then, after describing tracking error measures based on probability metrics, we propose tracking error portfolio selection problems consistent with tracking error orderings. In the last section we summarize our contribution.

2. Practical Reward-Risk Portfolio Problems Consistent with Risk Orderings

We begin this section by reviewing the concept of FORS orderings and measures and then propose portfolio problems where a benchmark asset with return r_Y and *n* risky assets with returns $r=[r_1,...,r_n]'$ are traded. No short selling is allowed; therefore the components x_i of the vector of weights $x=[x_1,x_2,...,x_n]'$ are non-negative. All portfolio problems in this section are consistent with the main risk orderings used in the literature and thus should serve to solve the first of the three portfolio problems identified in Section 1. Moreover, for all portfolio optimization problems, we assume to have *T* independent and identically distributed (i.i.d.) observations of the returns $r_{(k)}=[r_{1,k},...,r_{n,k}]'$ and of the benchmark return $r_{Y,k}$, k = 1,…,*T*.

According to utility theory, investors maximize an expected state-dependent utility function and thus they implicitly maximize their performance with respect to a given benchmark (see Rachev *et al.* (2008)). Thus, one general way to describe an ordering is by considering functionals on a product space $U=\Lambda\times V$ of joint random variables defined on $(\Omega,\Im,\Pr)$ with real values where Λ is a non-empty set of the admissible choices and *V* is the space of all possible benchmarks. However if the space of benchmarks is not mentioned, we assume $U=\Lambda$. Following Ortobelli *et al.* (2008), defining a FORS ordering necessitates a probability functional $\rho:U\times B\to\mathbb{R}$ (where (B,M_B) is a measurable space) that satisfies two properties: (1) the invariance in law and (2) the consistency of preferences. In particular, we refer to a *FORS measure induced by order of preference* $\succ$, any probability functional $\mu:U\to R$ that is consistent with respect to the order of preferences $\succ$. That is, if *X* is preferred to *Y* ($X\succ Y$), this implies that $\mu(X,Z)\le\mu(Y,Z)$ for a fixed and arbitrary benchmark *Z* belonging to *V*.

Let $\rho:U\times B\to\mathbb{R}$ (with $B=[\mathbf{a},\mathbf{b}]\subseteq\overline{\mathbb{R}}^m$, $-\infty\le a_i<b_i\le+\infty$, *i*=1,…,*m*) be a simple (invariant in law) functional (i.e., for every $X,Y\in\Lambda$, $\rho_X=\rho_Y\Leftrightarrow F_X=F_Y$) that is consistent with the order of preferences $\succ$ on the class Λ; that is, $\forall X,Y\in\Lambda$ $\rho_X\le\rho_Y$ anytime that *X* is preferred to *Y* with respect

to the order of preferences $\succ$ ($X \succ Y$). Then, we say *X dominates Y in the sense of FORS ordering induced by* $\succ$ (namely $X \underset{\succ}{FORS}\, Y$) if and only if $\rho_X(u) \le \rho_Y(u) \quad \forall u \in B$. We call the simple functional ρ, the FORS functional (measure) associated with the FORS ordering of random variables belonging to Λ. A FORS functional can satisfy different properties, so for example, we say that the FORS functional (measure) associated with the FORS ordering satisfies the convex property if for any $t \in B$ and $a \in [0,1]$, $\rho_{aX+(1-a)Y}(t) \le a\rho_X(t) + (1-a)\rho_Y(t)$. We call *FORS risk-ordering* (measure) any FORS ordering (measure) induced by (consistent with) the monotony order. A reward FORS measure v is a functional isotonic with respect to FORS-risk ordering induced $\succ$, i.e., if $X \underset{\succ}{FORS}\, Y$, then $v(X) \ge v(Y)$. Moreover, we say that X dominates Y with respect to a FORS-uncertainty order when $X \underset{\succ}{FORS}\, Y$ and $-X \underset{\succ}{FORS} -Y$ for some given FORS-risk orderings induced by the order of preferences $\succ$.

When $B = [a,b] \subseteq \overline{\mathbb{R}}$ and the FORS-risk measure associated with the FORS-risk ordering ρ_X is a bounded variation function for every random variable X belonging to Λ, then, for every $\alpha > 1$, we refer to α FORS-risk ordering induced by $\succ$ the following ordering defined:

$$X \underset{\succ,\alpha}{FORS}\ Y \text{ iff } \rho_{X,\alpha}(u) \le \rho_{Y,\alpha}(u) \quad \forall u \in [a,b],$$

where $\rho_{X,\alpha}(u) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^u (u-t)^{\alpha-1} d\rho_X(t) & \text{if } \alpha > 1 \\ \rho_X(u) & \text{if } \alpha = 1 \end{cases}$ is called *functional associated with the* α *FORS order* induced by $\succ$. For the fractional integral property, we can write

$$\rho_{X,\alpha}(t) = \frac{1}{\Gamma(\alpha - v)} \int_a^t (t-u)^{\alpha - v - 1} \rho_{X,v}(u) du. \tag{1}$$

Then, for every $\alpha > v \ge 1$, $X \underset{\succ,v}{FORS}\ Y$ implies $X \underset{\succ,\alpha}{FORS}\ Y$. Moreover, FORS orderings can be represented in terms of utility functions, since $X \underset{\succ,\alpha}{FORS}\, Y$ if and only if $\int_a^b \phi(u) d\rho_X(u) \ge \int_a^b \phi(u) d\rho_Y(u)$ for every ϕ belonging to a given class of functions W^α. When the simple probability FORS-risk measure $\rho_X(\lambda)$ associated with a FORS ordering for any $\lambda \in [a,b]$ is:

1) positively homogeneous (i.e., $\forall \alpha \geq 0 \;\; \rho_{\alpha X}(\lambda) = \alpha \rho_X(\lambda)$),

2) translation invariant $\forall t \in \mathbb{R}$, $\rho_{X+t}(\lambda) = \rho_X(\lambda) - t$),

3) sub-additive ($\rho_{X+Y}(\lambda) \leq \rho_X(\lambda) + \rho_Y(\lambda)$),

then $\rho_X(\lambda)$ is a coherent measure $\forall \lambda \in [a,b]$ in the sense of Artzner *et al.* (1999). In this case, we define ρ_X *coherent FORS functional* associated with the underlining ordering. Moreover, we call ρ_X *characteristic FORS functional* of the associated ordering when $\rho_X(\lambda)$ is only positively homogeneous and translation invariant $\forall \lambda \in [a,b]$. As suggested by Ortobelli *et al.* (2008), we can obtain a subclass of the optimal choices with respect to the order of preferences $\succ$ by solving the optimization problem:

$$\begin{aligned} &\min_{X \in \Lambda} \rho_X(t) \\ &\text{subject to } \; t \in B \end{aligned} \tag{2}$$

where $\rho : U \times B \to \mathbb{R}$ is a FORS measure associated with a FORS ordering induced by an order of preferences $\succ$. Since in many portfolio problems[4] investors employ different orderings of preferences for random rewards and random risks, then we could have a FORS-risk measure $\rho : U \times B_1 \to \mathbb{R}$ consistent with an ordering of the risks $\succ_1$ and a FORS-reward measure $v : U \times B_2 \to \mathbb{R}$ isotonic with an ordering of the rewards $\succ_2$. Thus, by minimizing the risk measure ρ provided that the expected reward v is constrained by some minimal value R for different values of $t_1 \in B_1; t_2 \in B_2$, as in the following portfolio problem:

$$\begin{aligned} &\min_{X \in \Lambda} \rho_X(t_1) \text{ subject to} \\ &v_X(t_2) \geq R; \\ &t_1 \in B_1; t_2 \in B_2, \end{aligned} \tag{3}$$

we obtain optimal choices consistent with a new ordering of preferences that take into account both orderings of rewards and risks. The importance of including the investor's preference toward reward in

[4] See Rachev *et al.* (2008)

portfolio analysis is well founded (see Rachev *et al.* (2008)). To consider both risk and reward, the so-called *performance measures* used in portfolio selection literature employ reward/risk ratios. Moreover, for some constraints and under some particular assumptions the solution of problem (3) is equivalent to the maximization of a reward/risk ratio v/ρ. As a matter of fact, according to Rachev *et al.* (2008), we know that when v is a positive, positively homogeneous, and concave FORS-reward measure induced by a risk ordering and ρ is a positive, positively homogeneous, and convex FORS measure that is consistent with another stochastic order, maximizing the ratio $v_X(t_2)/\rho_X(t_1)$ is equivalent to maximizing the measure v maintaining the risk ρ opportunely lower than a fixed risk. It is also equivalent to minimizing the risk ρ maintaining the measure v opportunely higher than a fixed reward. The following proposition summarizes these conditions:

Proposition 1 *Consider a frictionless economy where a benchmark asset with return* r_Y *and* $n \geq 2$ *risky assets with returns* $r = [r_1, ..., r_n]'$ *are traded. Let* v, ρ *be two probability functionals defined on a space of random portfolios with weights that belong to*

$$V = \left\{ x \in R^n / x'e = 1; Lb \leq Ax \leq Ub; A \in R^{n \times k}; Lb, Ub \in R^k \right\},$$

where we assume they are strictly positive. Suppose that v is a positively homogeneous concave FORS-reward measure induced by a risk ordering and ρ *is a positively homogeneous convex FORS-risk measure that is consistent with another stochastic order. If we maximize ratios* $v(x'r)/\rho(x'r)$, $v(x'r - r_Y)/\rho(x'r)$, *or* $v(x'r - r_Y)/\rho(x'r - r_Y)$ *subject to the portfolio weights that belong to the space* V, *we obtain non-dominated portfolios* $x'r$ *(or* $x'r - r_Y$*) with respect to both previous stochastic orders (of v and* ρ*).*

Proposition 1 justifies the use of performance measures that enable investors to determine optimal non-dominated choices. Furthermore, the portfolio selection problems based on reward-risk analysis of Proposition 1 define quasi-concave problems (see Rachev et al. (2008) and the references therein). In addition, from Bauerle and Müller (2006) we know that the minimization of a convex measure ρ that satisfies the Fatou property[5] gives a choice that cannot be dominated by the preferences of risk-averse investors (i.e., an alternative portfolio preferred to it by all risk-averse investors does not exist). Thus, we get the following corollary:

Corollary 1 *Under the assumption of Proposition 3, the solution to problem (3) and the maximization of ratios* $v(x'r)/\rho(x'r)$, $v(x'r-r_Y)/\rho(x'r)$, *or* $v(x'r-r_Y)/\rho(x'r-r_Y)$, *subject to the portfolio weights that belong to the convex closed space V, is a quasi-concave problem. Moreover, if the measures ρ and v satisfy the Fatou property, the optimal choices are optimal even for non-satiable risk-averse investors.*

There exist many possible generalizations to these results and there are many performance measures that do not fit the previous classification even if they present very good performance.[6] On the other hand, these considerations and the computational simplicity of the optimization problems suggest using optimization of the performance measures $v(x'r)/\rho(x'r)$ as an alternative to classical portfolio selection models. Next, we assume that the reward measure is the mean and we analyze mean-risk portfolio selection problems that are consistent with the most commonly used risk orderings in financial literature.

2.1 Practical Portfolio Selection Problems Associated with Stochastic Dominance Orderings

Recall that *X* dominates *Y* with respect to α stochastic dominance order $X \underset{\alpha}{\geq} Y$ (with $\alpha \geq 1$) if and only if $E(u(X)) \geq E(u(Y))$ for all *u* belonging to a given class U_α of utility functions.[7] Moreover the derivatives of *u* satisfy the inequalities $(-1)^{k+1}u^{(k)} \geq 0$ where $k = 1,\ldots, m-1$ for the integer *m* that satisfies $m-1 \leq \alpha < m$. The ordering $X \underset{\alpha}{\geq} Y$ is also equivalent to saying that for every real *t*:

$$F_X^{(\alpha)}(t) := E\left(\left(t-X\right)_+^{\alpha-1}\right)\Big/\Gamma(\alpha) \leq F_Y^{(\alpha)}(t) \text{ when } \alpha>1, \text{ and } F_X(t) = \Pr(X \leq t) \leq F_Y(t) = \Pr(Y \leq t) \text{ when } \alpha=1, \quad (4)$$

where $\Gamma(\alpha) = \int_0^{+\infty} z^{\alpha-1}e^{-z}dz$. In particular, first-degree stochastic dominance (FSD) states an ordering of

[5] That is, for any sequence $\{X_n\}_{n\in\mathbb{N}}$ of integrable random variables such that $E\left(\left|X_n - X\right|\right) \xrightarrow{n\to\infty} 0$, implies $\rho(X) \leq \lim_{k\to\infty} \inf_{n\geq k} \rho(X_n)$.
[6] See, for example, the Rachev ratio and generalized Rachev ratio in Rachev *et al.* (2008).
[7] See Levy (1992) and Ortobelli et al. (2008) and the references therein..

preferences for non-satiable agents and second-degree stochastic dominance (SSD) states an ordering of preferences for non-satiable risk-averse investors. Moreover, we refer to α bounded stochastic dominance order between X and Y (namely, $X \underset{\alpha}{\overset{b}{\geq}} Y$) when $F_X^{(\alpha)}(t) \leq F_Y^{(\alpha)}(t)$ for any t belonging to the support of X and Y. Consider T i.i.d. observations of the returns $r = [r_1, \ldots, r_n]'$ and of the benchmark return r_Y. Then, a consistent estimator of $\Gamma(\alpha) F_{x'r-r_Y}^{(\alpha)}(t)$ is given by $\hat{G}_{x'r-r_Y}^{(\alpha)}(t) = \frac{1}{T}\sum_{k=1}^{T}\left(t - x' r_{(k)} + r_{Y,k}\right)^{\alpha-1} I_{[t > x' r_{(k)} - r_{Y,k}]}$ where $I_{[t > x' r_{(k)} - r_{Y,k}]} = \begin{cases} 1 & \text{if } t > x' r_{(k)} - r_{Y,k} \\ 0 & \text{otherwise} \end{cases}$. When no short sales are allowed, the support of all admissible portfolios is given by the interval $\left(\min_{x\in S} \min_{1\leq k\leq T} x' r_{(k)} - r_{Y,k}, \max_{x\in S} \max_{1\leq k\leq T} x' r_{(k)} - r_{Y,k}\right)$ where $S = \left\{x \in \mathbb{R}^n / \sum_{i=1}^n x_i = 1; x_i \geq 0\right\}$. However, for all $t \in \left(\min_{x\in S} \min_{1\leq k\leq T} x' r_{(k)} - r_{Y,k}, \max_{x\in S} \min_{1\leq k\leq T} x' r_{(k)} - r_{Y,k}\right)$, minimizing $\hat{G}_{x'r-r_Y}^{(\alpha)}(t)$ gives the value 0 and the optimal weights $\hat{x} = \arg\left(\max_{x\in S} \min_{1\leq k\leq T} x' r_{(k)} - r_{Y,k}\right)$. In order to find portfolios that are not first-order dominated, we should minimize $\hat{G}_{x'r}^{(1)}(t) = \frac{1}{T}\sum_{k=1}^{T} I_{[t > x' r_{(k)} - r_{Y,k}]}$ for any $t \geq c = \max_{x\in S} \min_{1\leq k\leq T} x' r_{(k)} - r_{Y,k}$. Instead, in order to find in mean-risk space the optimal portfolios that are non-dominated with respect to α stochastic dominance order for $\alpha > 1$ for a given mean equal to or greater than m and a parameter $t \geq c = \max_{x\in S} \min_{1\leq k\leq T} x' r_{(k)} - r_{Y,k}$, one solves the following optimization problem with linear constraints:

$$\begin{aligned} &\min_x \frac{1}{T}\sum_{k=1}^{T} v_k^{\alpha-1} \text{ subject to} \\ &\frac{1}{\mathrm{T}}\sum_{t=1}^{T} x' r_{(t)} \geq m; \;\; \sum_{j=1}^{n} x_j = 1; \;\; x_j \geq 0; \;\; j = 1, \ldots, n \\ &v_k \geq 0; \;\; v_k \geq t + r_{Y,k} - x' r_{(k)}, \;\; k = 1, \ldots, T \quad . \end{aligned} \tag{5}$$

In particular, in order to get optimal choices for non-satiable risk-averse investors, we solve the previous linear programming (LP) problem for $\alpha = 2$, which is then a convex optimization problem that is linearizable using the fractional integral property (1). As a matter of fact, for $\alpha > 2$,

$\tilde{G}^{(\alpha)}_{x'r-r_Y}(t) \xrightarrow{M,T\to\infty} \Gamma(\alpha-2)F^{(\alpha)}_{x'r-r_Y}(t) = \int_{-\infty}^{t}(t-u)^{\alpha-3}E\left(\left(u-x'r+r_Y\right)_+\right)du$ for any $t \ge c = \max_{x\in S}\min_{1\le k\le T} x'r_{(k)} - r_{Y,k}$

where $\tilde{G}^{(\alpha)}_{x'r-r_Y}(t) = \frac{t-c}{M}\sum_{i=1}^{M-1}\left((M-i)\left(\frac{t-c}{M}\right)\right)^{\alpha-3}\frac{1}{T}\sum_{k=1}^{T}\left(c+i\left(\frac{t-c}{M}\right)-x'r_{(k)}+r_{Y,k}\right)^{\alpha-1}I_{\left[c+i\left(\frac{t-c}{M}\right)>x'r_{(k)}-r_{Y,k}\right]}$. Thus,

$\tilde{G}^{(\alpha)}_{x'r-r_Y}(t)$ is a consistent estimator of $\Gamma(\alpha-2)F^{(\alpha)}_{x'r-r_Y}(t)$ when M is large enough[8]. So, in a mean-risk space, we get non-dominated portfolios with respect to α ($\alpha > 2$) stochastic dominance bounded order by solving the following LP problem for $t \ge c = \max_{x\in S}\min_{1\le k\le T} x'r_{(k)} - r_{Y,k}$, and a mean equal to or greater than *m*:

$$\begin{aligned}
&\min_x \frac{t-c}{M}\sum_{i=1}^{M-1}\left((M-i)\left(\frac{t-c}{M}\right)\right)^{\alpha-3}\frac{1}{T}\sum_{i=1}^{T}v_{k,i} \text{ subject to} \\
&\frac{1}{\mathrm{T}}\sum_{t=1}^{T}x'r_{(t)} \ge m;\ \sum_{j=1}^{n}x_j = 1;\ x_j \ge 0;\ j=1,\dots,n;\ v_{k,i} \ge 0; \\
&v_{k,i} \ge c + i\left(\frac{t-c}{M}\right) - x'r_{(k)} + r_{Y,k},\ k=1,\dots,T; i=1,\dots,M.
\end{aligned} \tag{6}$$

2.2 Practical Portfolio Problems Associated with Inverse Stochastic Dominance Orderings

As an alternative to classic stochastic orders, we can use the dual (also called inverse) representations of stochastic dominance rules.[9] We say *X* dominates *Y* with respect to α inverse stochastic dominance order $X \underset{-\alpha}{\ge} Y$ (with $\alpha \ge 1$) if and only if for every $p \in [0,1]$,

$$F_X^{(-\alpha)}(p) = \frac{1}{\Gamma(\alpha)}\int_0^p (p-u)^{\alpha-1}dF_X^{-1}(u) \ge F_Y^{(-\alpha)}(p) \text{ when } \alpha > 1, \text{ and } F_X^{-1}(p) \ge F_Y^{-1}(p) \text{ when } \alpha = 1$$

where $F_X^{-1}(0) = \lim_{p\to 0}F_X^{-1}(p)$ and $F_X^{-1}(p) = \inf\{x : \Pr(X \le x) = F_X(x) \ge p\}$ $\forall p \in (0,1]$, is the left inverse of the cumulative distribution function F_X. Thus in this case, $-F_X^{(-\alpha)}(p)$ is the FORS measure associated with this FORS-risk ordering. In risk management literature, the opposite of the *p*-quantile $F_X^{-1}(p)$ of *X*

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[8] We should choose *M* equal to or larger than $(t-c)/d$ where $d = \min_x \inf\left\{z = \left(\left(x'r-r_Y\right)_{k:T} - \left(x'r-r_Y\right)_{k-1:T}\right) \middle| z > 0; k = 2,\dots,T\right\}$ and $\left(x'r-r_Y\right)_{k:T}$ points out the k-*th* observation of *T* ordered observations of $\left(x'r-r_Y\right)$.

[9] See Ortobelli et al. (2008) and the references therein.

is referred to as value-at-risk (VaR) of *X,* namely $VaR_p(X) = -F_X^{-1}(p)$. VaR expresses the maximum loss among the best 1-*p* percentage cases that could occur for a given horizon. A consistent statistic of the *p*-quantile of *X* is given by the order statistic $X_{[pT]:T}$ that indicates the [*pT*]-*th* observation of *T* ordered observations of *X*. Then, in order to find the optimal portfolio that is non-dominated at the first inverse order, we minimize $-X_{[pT]:T}$ under usual constraints. Moreover, using the fractional integral property (1) for any $\alpha > 1$, we get $\Gamma(\alpha-1)F_X^{(-\alpha)}(p) = \int_0^p (p-u)^{\alpha-2} F_X^{-1}(u)du$. Assuming equally probable *T* scenarios, a consistent estimator of $\Gamma(\alpha-1)F_{x'r-r_Y}^{(-\alpha)}(p)$ when $p = s/T$ for a given integer $s \le T$ is given by $\frac{1}{T}\sum_{i=1}^{s-1}\left(\frac{s-i}{T}\right)^{\alpha-2}(x'r-r_Y)_{i:T}$. Thus, we get portfolios in a mean-risk space that are optimal with respect to the α-th inverse stochastic dominance order for any $\alpha > 1$ by solving the optimization problem:

$$\begin{aligned} &\min_x \; -\sum_{i=1}^{s-1}\left(\frac{s-i}{T}\right)^{\alpha-2}(x'r-r_Y)_{i:T} \quad \text{subject to} \\ &\frac{1}{\mathrm{T}}\sum_{t=1}^{T} x'r_{(t)} \ge m; \quad \sum_{j=1}^{n} x_j = 1; x_j \ge 0; \; j = 1,\dots,n \end{aligned} \tag{7}$$

for a given $s \le T$ and a mean equal to or greater than *m.* Recall that first- and second-stochastic dominance rules are equivalent to their inverse orderings.[10] Furthermore, $F_X^{(-2)}(p) = L_X(p) = \int_0^p F_X^{-1}(t)dt$ is the absolute Lorenz curve (or absolute concentration curve) of asset *X* with respect to its distribution function F_X. The absolute concentration curve $L_X(p)$ valued at *p* shows the mean return accumulated up to the lowest *p* percentage of the distribution. Both measures and $L_X(p)$ have important financial and economic interpretations and are widely used in the recent risk literature. In particular, the negative absolute Lorenz curve divided by probability *p* is a coherent risk measure in the sense of Artzner *et al.* (1999) that is called conditional value-at-risk (CVaR) or expected

[10] That is, $X \; FSD \; Y \Leftrightarrow F_X^{-1}(p) \ge F_Y^{-1}(p), \quad p \in [0,1]$ and $X \; SSD \; Y \Leftrightarrow F_X^{(-2)}(p) \ge F_Y^{(-2)}(p), \quad p \in [0,1]$.

shortfall, expressed as

$$CVaR_p(X) = -L_X(p)/p = \inf_u \left\{ u + E\left((-X-u)_+\right)/p \right\} \tag{8}$$

where the optimal value u is $VaR_p(X) = -F_X^{-1}(p)$. Thus, as a consequence of Equation (8), the recent literature has shown that we can minimize CVaR for a fixed mean by solving a LP problem. Moreover, coherent risk measures using specific functions for the Lorenz curve are easily obtained. In particular, we observe that some classic Gini-type measures are coherent measures. As a matter of fact, by definition, for every $v \geq 1$ and for every $\beta \in (0,1)$ we have that

$$GT_{(\beta,v)}(X) := -\Gamma(v+1) F_X^{(-(v+1))}(\beta)/\beta^v = -(v-1)v\int_0^\beta (\beta-u)^{v-2} L_X(u)du \Big/ \beta^v$$

is consistent with $\underset{-(v+1)}{\geq}$. Then using the coherency of CVaR, we easily prove that $GT_{(\beta,v)}(X)$ is a coherent FORS functional associated with (v+1) inverse stochastic dominance order and the following remark holds.

Remark 1 *For every $v \geq 1$ and for every $\beta \in (0,1)$, the measure $GT_{(\beta,v)}(X) = -\Gamma(v+1)F_X^{(-(v+1))}(\beta)/\beta^v$ is a linearizable coherent risk measure consistent with $\underset{-(v+1)}{\geq}$ order.*

Since the measures $GT_{(\beta,v)}(X)$ are strictly linked to the extended Gini mean difference (as we will show in the next section), we refer to them as Gini-type coherent measures. Moreover, from Remark 1, when we minimize the measure $GT_{(\beta,v)}(x'r - r_Y)$ and $\beta = s/T$ for a given integer $s \leq T$, then using Equation (8) we can linearize its consistent estimator $\frac{-v(v-1)}{\beta^v T}\sum_{i=1}^{s-1}\left(\frac{s-i}{T}\right)^{v-2}\hat{L}_{x'r-r_Y}\left(\frac{i}{T}\right)$ where $\hat{L}_{x'r-r_Y}\left(\frac{i}{T}\right) = \frac{1}{T}\sum_{k=1}^{i}(x'r-r_Y)_{k:T}$.

In particular, we can determine choices that are optimal with respect to α inverse stochastic dominance orders for any $\alpha > 2$ by solving the LP problem:

$$\begin{aligned}&\min_{x,b_1,\dots,b_{s-1}} \sum_{i=1}^{s-1}\left(\frac{s-i}{T}\right)^{\alpha-3}\left(\frac{i}{T}b_i+\frac{1}{T}\sum_{t=1}^{T}v_{t,i}\right) \text{ subject to}\\ &\frac{1}{\mathrm{T}}\sum_{t=1}^{T}x'r_{(t)}\geq m;\ \sum_{j=1}^{n}x_j=1; x_j\geq 0;\ j=1,\dots,n;\ v_{t,i}\geq 0;\\ &v_{t,i}\geq -x'r_{(t)}+r_{Y,t}-b_i;\ t=1,\dots,T; i=1,\dots,s-1\quad ,\end{aligned}\tag{9}$$

for a given mean equal to or greater than *m.*

Other typical coherent FORS functionals associated with FORS orderings can be derived from Acerbi's (2002) spectral measures. Hence, any spectral measure

$$M_{\phi}(X)=-\int_0^1\phi(u)F_X^{-1}(u)du\tag{10}$$

is a coherent risk measure identified by its risk spectrum ϕ that is an *a.e.* non-negative decreasing and integrable function such that $\int_0^1\phi(u)du=1$. Moreover, for any spectrum ϕ with $\phi>0,\ \phi'<0,$ in (0,1), we can introduce a new ordering, that we call ϕ*-spectral FORS-risk ordering*, so that we say that *X* dominates *Y* with respect to the ϕ-spectral FORS-risk ordering (namely, $X\ \phi\ \underset{\geq}{FORS}\ Y$) if and only if

$$ST_{(\phi,X)}(t):=\frac{-1}{\int_0^t\phi(u)du}\int_0^t\phi(u)F_X^{-1}(u)du\leq ST_{(\phi,Y)}(t)\quad \forall t\in(0,1]\tag{11}$$

This is a FORS-risk ordering, since it is consistent with monotone ordering and $ST_{(\phi,Y)}=ST_{(\phi,X)}$ if and only if $F_X=F_Y$ .Moreover, for any $t\in(0,1)$, $ST_{(\phi,X)}(t)$ is a spectral coherent measure with spectrum

$$\tilde{\phi}(u)=\begin{cases}\dfrac{\phi(u)}{\int_0^t\phi(v)dv} & \text{if } u\leq t\\ 0 & \text{if } u\in(t,1]\end{cases}.\tag{12}$$

Furthermore, we can rewrite the spectral FORS measure associated with the ϕ-spectral FORS ordering as:

$$ST_{(\phi,X)}(\beta)=\frac{-1}{\int_0^\beta\phi(u)du}\int_0^\beta\phi(u)F_X^{-1}(u)du=\frac{1}{\int_0^\beta\phi(u)du}\int_0^\beta\phi'(u)L_X(u)du-\frac{\phi(\beta)}{\int_0^\beta\phi(u)du}L_X(\beta)\quad\forall\beta\in(0,1].\tag{13}$$

Again, we can apply Equation (8) to linearize the concentration curve of Equation (13), since $\phi'<0$. Thus assuming $\beta=s/T$ for a given integer $s\leq T$, we can use the consistent estimator of $ST_{(\phi,x'r-r_Y)}(\beta)$:

$$\frac{1}{\int_0^\beta \phi(u)du}\left(\frac{1}{T}\sum_{i=1}^{s-1}\phi'\left(\frac{i}{T}\right)\hat{L}_{x'r-r_Y}\left(\frac{i}{T}\right)-\phi(\beta)\hat{L}_{x'r-r_Y}(\beta)\right).$$

We obtain the optimal choices with respect to ϕ-spectral FORS-risk ordering by solving the following LP problem:

$$\begin{aligned}
&\min_{x,\alpha_1,\ldots,a_{T-1},b}\phi(\beta)\left(b\beta+\frac{1}{T}\sum_{k=1}^{T}u_k\right)-\frac{1}{T}\sum_{i=1}^{s-1}\phi'\left(\frac{i}{T}\right)\left(\frac{i}{T}a_i+\frac{1}{T}\sum_{k=1}^{T}v_{k,i}\right) \quad \text{subject to}\\
&v_{t,i}\geq -x'r_{(t)}+r_{Y,t}-a_i;\ v_{t,i}\geq 0;\ \frac{1}{\mathrm{T}}\sum_{t=1}^{T}x'r_{(t)}\geq m;\ t=1,\ldots,T;i=1,\ldots,s-1 \qquad (14)\\
&\sum_{i=1}^{n}x_j=1;\ \ x_j\geq 0;\ \ j=1,\ldots,n;u_t\geq 0;u_t\geq -b-x'r_{(t)}+r_{Y,t} \quad ,
\end{aligned}$$

for some given mean equal to or greater than *m.*

2.3 Practical Portfolio Selection Problems Associated with Behavioral Finance Orderings

The first behavioral finance orderings introduced in the finance literature are the orderings deduced from Markowitz' studies on investors' utility (referred to as *Markowitz orderings*) and the orderings according to prospect theory (referred to as *Prospect orderings*)[11]. Optimal choices consistent with these orderings are optimal for non-satiable investors that are neither risk-averse nor risk-lovers. According to the definition of prospect ordering given by Levy and Levy (2002), *X* dominates *Y* in the sense of prospect theory ($X \ \ P\,SD\ \ Y$) if and only if

$$\forall(a,y)\in[0,1]\times(-\infty,0],\ g_X(a,y):=ag_X(y)+(1-a)\tilde{g}_X(y)\leq g_Y(a,y),$$

where $g_X(y):=\int_0^{-y}F_X(u)du=E\left(\left(-y-X\right)I_{[X\in(0,-y]]}\right)-yF_X(0)$ and $\tilde{g}_X(y):=\int_y^0 F_X(u)du=-E\left(\left(X-y\right)I_{[X\in(y,0]]}\right)-yF_X(0)$.

Since a consistent estimator of $g_X(a,y)$ is given by

$$\hat{g}_{x'r-r_Y}(a,y)=\frac{1}{T}\left(a\sum_{k=1}^{T}\left(-y-x'r_{(k)}+r_{Y,k}\right)I_{[x'r_{(k)}-r_{Y,k}\in(0,-y]]}-y\sum_{k=1}^{T}I_{[x'r_{(k)}-r_{Y,k}<0]}-(1-a)\sum_{k=1}^{T}\left(x'r_{(k)}-r_{Y,k}-y\right)I_{[x'r_{(k)}-r_{Y,k}\in(y,0]]}\right),$$

we get non-dominated portfolios with respect to prospect theory order minimizing $\hat{g}_{x'r-r_Y}(a,y)$ for

[11] See Levy and Levy (2002) and the references therein..

different values of $(a,y)\in[0,1]\times[c,0]$, where $c=-\max\left(\left|\min_{x\in S}\min_{1\le k\le T} x' r_{(k)}-r_{Y,k}\right|,\left|\max_{x\in S}\max_{1\le k\le T} x' r_{(k)}-r_{Y,k}\right|\right)$ and $S=\left\{x\in\mathbb{R}^n / \sum_{i=1}^n x_i=1; x_i\ge 0\right\}$.

According to the definition of Markowitz orderings given by Levy and Levy (2002), we say *X dominates Y in the sense of Markowitz order* ($X\ M\ SD\ Y$) if and only if for every $y\in(-\infty,0]$

$$m_X(y):=\int_{-\infty}^{y} F_X(u)du=F_X^{(2)}(y)=E\left(\left(y-X\right)_+\right)\le m_Y(y) \text{ and } \int_{-y}^{+\infty} F_Y(u)-F_X(u)du=E\left(\left(X+y\right)_+\right)-E\left(\left(Y+y\right)_+\right)\ge 0$$

if and only if

$$m_X(a,y):=a m_X(y)-(1-a)E\left(\left(X+y\right)_+\right)\le m_Y(a,y),\ \forall(a,y)\in[0,1]\times(-\infty,0]$$

In this case, we obtain optimal portfolios in the sense of Markowitz order by solving the following portfolio problem in a mean-risk space:

$$\begin{aligned}&\min_x \frac{a}{T}\sum_{k=1}^{T} v_k-\frac{1-a}{T}\sum_{k=1}^{T}\left(x' r_{(k)}-r_{Y,k}+y\right)I_{[x' r_{(k)}-r_{Y,k}>-y]} \text{ subject to}\\ &\frac{1}{T}\sum_{t=1}^{T} x' r_{(t)}\ge m;\ \sum_{j=1}^{n} x_j=1;\ x_j\ge 0;\ j=1,\dots,n;\\ &v_k\ge 0;\ v_k\ge y+r_{Y,k}-x' r_{(k)}; k=1,\dots,T\qquad ,\end{aligned}\tag{15}$$

for different values of $(a,y)\in[0,1]\times[c,0]$ with $c=-\max\left(\left|\min_{x\in S}\min_{1\le k\le T} x' r_{(k)}-r_{Y,k}\right|,\left|\max_{x\in S}\max_{1\le k\le T} x' r_{(k)}-r_{Y,k}\right|\right)$ and a given mean equal to or greater than *m.* Moreover, we can propose portfolio selection problems consistent with the generalized Markowitz- and Prospect-type orderings proposed by Ortobelli *et al.* (2008). In addition, we introduce many new kinds of behavioral finance orderings associated with the FORS *aggressive-coherent functionals*

$$S_X(a,t_1,t_2)=a\rho_{1,X}(t_2)-(1-a)\rho_{2,-X}(t_1)$$

where $a\in[0,1]$, $t_1\in B_1, t_2\in B_2$ and $\rho_{1,X},\rho_{2,X}$ are two simple coherent FORS functionals.[12] Even if we minimize one of these FORS aggressive-coherent functionals, we obtain optimal choices for non-satiable investors that are neither risk-averse nor risk-lovers. For example, for any $(a,t,\beta)\in[0,1]\times[0,1]\times[0,1]$ we

[12] See Rachev *et al.* (2008) and Ortobelli *et al.* (2008).

can consider either the Gini type functional $g_X(a,t,\beta)=aGT_{(t,v_1)}(X)-(1-a)GT_{(\beta,v_2)}(-X)$ where $v_1,v_2>1$. For this functional, we derive behavioral finance orderings that are strongly dependent on the absolute Lorenz concentration curve. Thus, the associated portfolio problems can be partially linearized using Equation (8). As a matter of fact, when $\beta=s/T$ and $t=k/T$ for some given integers $s,k\le T$, the consistent estimator of $g_{x'r-r_Y}(a,t,\beta)$ is given by:

$$\hat{g}_{x'r-r_Y}(a,t,\beta)=\frac{-av_1(v_1-1)}{\beta^{v_1}T}\sum_{i=1}^{s-1}\left(\frac{s-i}{T}\right)^{v_1-2}\hat{L}_{x'r-r_Y}\left(\frac{i}{T}\right)+\frac{(1-a)v_2(v_2-1)}{\beta^{v_2}T}\sum_{i=1}^{k-1}\left(\frac{k-i}{T}\right)^{v_2-2}\hat{L}_{r_Y-x'r}\left(\frac{i}{T}\right).$$

If we minimize this functional we can apply Equation (8) to linearize the concentration curve $-\hat{L}_{x'r-r_Y}(i/T)$. However, the second concentration curve $\hat{L}_{r_Y-x'r}(i/T)$ should be estimated using $\frac{1}{T}\sum_{k=1}^{i}(r_Y-x'r)_{k:T}$ since, in the minimization problem, it appears with a positive sign. The high computational complexity of these portfolio selection models is a common problem for all previously proposed models which are consistent with behavioral finance orderings.

3. Practical Portfolio Problems with Uncertainty and Tracking Error Orderings

In this section, we present portfolio selection problems where investors want to optimize the uncertainty (as in example (2) in Section 1) or where investors want to track a benchmark as closely as possible (like in example (3) in Section 1). Moreover, we clarify some connections between the theory of orderings and the theory of probability distances/metrics (see Rachev (1991)).

3.1 Practical Portfolio Problems Consistent with Uncertainty Orderings

Probably the most well known uncertainty ordering in financial literature is the concave ordering more popularly referred to as the Rothshild-Stiglitz (R-S) ordering.[13] We say that *X* dominates *Y* in the sense of Rothschild and Stiglitz (X RSD Y) if and only if all risk-averse investors prefer the less uncertain

[13] See Levy (1992) and the references therein.

variable X to Y (if and only if E(X) = E(Y) and $X \underset{2}{\geq} Y$). More generally, we state that X dominates Y in the sense of α_1,α_2 - *R-S* order (with $\alpha_1,\alpha_2 \geq 2$) if and only if $X \underset{\alpha_1}{\geq} Y$ and $-X \underset{\alpha_2}{\geq} -Y$, that is, if and only if

$$F_X^{(\alpha_1,\alpha_2)}(b,t) := bF_X^{(\alpha_1)}(t) + (1-b)\overline{F}_X^{(\alpha_2)}(t) \leq F_Y^{(\alpha_1,\alpha_2)}(b,t) \text{ for every } b \in [0,1], t \in \mathbb{R} \,, \tag{16}$$

where $\overline{F}_X^{(\alpha)}(t) = E\left(\left(X-t\right)_+^{\alpha-1}\right)\Big/\Gamma(\alpha) = F_{-X}^{(\alpha)}(-t) = \dfrac{1}{\Gamma(\alpha-2)}\int_{-\infty}^{-t}(-t-u)^{\alpha-3}E\left(\left(X+u\right)_+\right)du$. Generally, we refer to α_1,α_2 - *R-S* as simply α -*R-S* order when $\alpha = \alpha_1 = \alpha_2$. Similarly, we state X dominates Y in the sense of α_1,α_2 -inverse R-S order (with $\alpha_1,\alpha_2 \geq 2$) if and only if, $X \underset{-\alpha_1}{\geq} Y$ and $-X \underset{-\alpha_2}{\geq} -Y$, that is, if and only if

$$F_X^{(-\alpha_1,-\alpha_2)}(b,p) := bF_X^{(-\alpha_1)}(p) + (1-b)\overline{F}_X^{(-\alpha_2)}(p) \geq F_Y^{(-\alpha_1,-\alpha_2)}(b,p) \quad \forall b \in [0,1], \forall p \in [0,1] \,, \tag{17}$$

where $\overline{F}_X^{(-\alpha)}(p) = \dfrac{1}{\Gamma(\alpha)}\int_p^1 (u-p)^{\alpha-1} dF_X^{-1}(u) = F_{-X}^{(-\alpha)}(1-p) = \dfrac{1}{\Gamma(\alpha-2)}\int_0^{1-p}(1-p-u)^{\alpha-3}L_{-X}(u)du$. When $\alpha_1 = \alpha_2 = 2$ in the order relations (16) and (17), we get the classic R-S order. Clearly, changing the parameters in (16) and (17), we can obtain many alternative uncertainty measures consistent with the respective R-S type ordering. For example,

$$F_X^{(3,3)}\left(0.5, E(X)\right) = \frac{E\left(\left(E(X)-X\right)_+^2\right) + E\left(\left(X-E(X)\right)_+^2\right)}{4} = \frac{1}{2}\left(\int_{-\infty}^{E(X)} E\left(\left(t-X\right)_+\right)dt + \int_{-\infty}^{-E(X)} E\left(\left(X+t\right)_+\right)dt\right)$$

gives a quarter of the variance of X that is obviously an uncertainty measure. Since for all these orderings we assume $\alpha_1,\alpha_2 \geq 2$, the associated functionals satisfy the convexity property and they can be easily linearized when we minimize their consistent estimators of $F_X^{(\pm\alpha_1,\pm\alpha_2)}(b,t)$ as suggested in Section 2. For example, as a consequence of the results in Section 2.1, we can minimize the variance for a mean equal to or greater than a value *m,* solving the classic Markowitz' mean variance problem with the following LP optimization problem:

$$\min_x \frac{\sum_{t=1}^{T} x' r_{(t)} - Tc}{TM} \sum_{i=1}^{M-1} \frac{1}{T} \sum_{k=1}^{T} v_{k,i} + \frac{-\sum_{t=1}^{T} x' r_{(t)} - Tc}{TM_1} \sum_{i=1}^{M_1-1} \frac{1}{T} \sum_{k=1}^{T} u_{k,i} \text{ subject to}$$
$$\frac{1}{\mathrm{T}} \sum_{t=1}^{T} x' r_{(t)} \geq m;\ \sum_{j=1}^{n} x_j = 1;\ \ x_j \geq 0;\ j = 1,...,n; v_{k,i} \geq c + i \frac{1}{M} \left(\frac{1}{\mathrm{T}} \sum_{t=1}^{T} x' r_{(t)} - c \right) - x' r_{(k)}; \tag{18}$$
$$u_{k,i} \geq c + i \frac{1}{M_1} \left(-c - \frac{1}{\mathrm{T}} \sum_{t=1}^{T} x' r_{(t)} \right) + x' r_{(k)};\ u_{k,i} \geq 0;\ v_{k,i} \geq 0;\ k = 1,...,T;$$

where $c = -\max\left(\left|\min_{x\in S}\min_{1\leq k\leq T} x' r_{(k)} - r_{Y,k}\right|, \left|\max_{x\in S}\max_{1\leq k\leq T} x' r_{(k)} - r_{Y,k}\right|\right)$, and M, M_1 are large. However, we also suggest the use of an uncertainty measure for some particular problems such as problem (2) in Section 1. Let us apply the use of uncertainty measures in that case and assume we want optimize the choice among n European options on n indexes with log-returns $z = [z_1,...,z_n]'$. Considering T i.i.d. observations of the vector of log-returns $z_{(k)} = [z_{1,k},...,z_{n,k}]'$ indexes k=1,…,*T,* then we can apply the Taylor approximation of the derivatives returns $R_{i,t} = \left(V_{i,t+1} - V_{i,t}\right) / V_{i,t}$ obtaining for historical data the quadratic relation $R_{i,t} = Ar_{i,t}^2 + Br_{i,t} + C$, where $A = P_{i,t}^2 \Gamma / 2V_{i,t}$, $B = P_{i,t} \Delta / V_{i,t}$, $C = \Theta / V_{i,t}$ are functions of the Greeks $\Gamma = \frac{\partial^2 V_{i,t}}{\partial P_{i,t}^2}$, $\Delta = \frac{\partial V_{i,t}}{\partial P_{i,t}}$ and $\Theta = \frac{\partial V_{i,t}}{\partial t}$, while $V_{i,t}$, $r_{i,t} = \left(P_{i,t+1} - P_{i,t}\right) / P_{i,t}$, and $P_{i,t}$ are, respectively, the value of the i-th European option, the return of the underlying i-th asset for the option, and the spot price of the underlying i-th asset at time t. Therefore, if we have T i.i.d. observations of the vector of log-returns $z_{(t)} = [z_{1,t},...,z_{n,t}]'$, t =1,…,*T*, we can easily obtain T i.i.d. observations of the returns $r_{i,t} = \exp(z_{i,t}) - 1$. Using the above transformation, we get T i.i.d. "approximated" observations of the vector of option returns, $R_{(t)} = [R_{1,t},...,R_{n,t}]'$. Since option return distributions present asymmetries and heavy tails, we can control them considering opportunely[14] the first four moments of the option returns. Then, investors could determine optimal choices isotonic with α_1, α_2- *R-S* order of the log returns by solving the following optimization problem for some values *m,* q_i; $i = 1,2$ and *t*:

[14] There is a strong connection between moments and stochastic orders as many authors have pointed out (see, among others, Levy (1992) and Ortobelli *et al*. (2008) and the references therein.

$$\begin{aligned}&\max_{x}\ \frac{b}{T}\sum_{k=1}^{T}\left(t-x'z_{(k)}\right)^{\alpha_1-1}I_{[t-x'z_{(k)}\geq 0]}+\frac{1-b}{T}\sum_{k=1}^{T}\left(x'z_{(k)}-t\right)^{\alpha_2-1}I_{[x'z_{(k)}-t\geq 0]}\\&\text{subject to }\frac{E\left(\left(x'R-E(x'R)\right)^3\right)}{\left(x'Q_Rx\right)^{3/2}}\geq q_1;\ \frac{E\left(\left(x'R-E(x'R)\right)^4\right)}{\left(x'Q_Rx\right)^{2}}\leq q_2\\&\frac{1}{T}\sum_{k=1}^{T}x'R_{(k)}\geq m\,x'Q_Rx;\ \sum_{j=1}^{n}x_j=1;\ \ x_j\geq 0;\ \ j=1,...,n;\end{aligned}\tag{19}$$

where Q_R is the variance-covariance matrix of the vector of option returns $R=[R_1,...,R_n]'$. For this optimization problem, we maximize the uncertainty of portfolio of log returns requiring that the Sharpe performance of option returns is equal to or greater than *m* for some opportune skewness and kurtosis.

Alternatively we can solve the problem associated with the α_1,α_2-inverse *R-S* order ($\alpha_1,\alpha_2>2$) solving the analogous problem for some values *m,* q_i; $i=1,2$ and $p=s/T$ (for an integer $s\leq T$)

$$\begin{aligned}&\max_{x,a_i,c_i}\ \frac{b}{T}\sum_{i=1}^{s-1}\left(\frac{s-i}{T}\right)^{\alpha_1-1}(x'z)_{i:T}+\frac{1-b}{T}\sum_{k=1}^{T-s-1}\left(\frac{T-s-k}{T}\right)^{\alpha_2-1}(-x'z)_{k:T}\\&\text{subject to }\frac{E\left(\left(x'R-E(x'R)\right)^3\right)}{\left(x'Q_Rx\right)^{3/2}}\geq q_1;\ \frac{E\left(\left(x'R-E(x'R)\right)^4\right)}{\left(x'Q_Rx\right)^{2}}\leq q_2\\&\frac{1}{T}\sum_{k=1}^{T}x'R_{(k)}\geq m\,x'Q_Rx;\ \sum_{j=1}^{n}x_j=1;\ \ x_j\geq 0;\ \ j=1,...,n;\end{aligned}\tag{20}$$

where $\left(x'z\right)_{k:T}$ is the *k*-th of *T* ordered observations of the underlying log returns portfolio $x'z$. Moreover, in many portfolio selection problems some concentration measures have been used to measure the uncertainty of the choices. The classical example is Gini's mean difference (GMD) and its extensions related to the fundamental work of Gini[15].

3.1.1 Gini's Mean Difference and Extensions

Gini's mean difference (GMD) is twice the area between the absolute Lorenz curve and the line joining the origin with the mean located on the right boundary vertical. We report here the most used of the myriad representations:

$$\Gamma_X(2)=E(X)-2\int_0^1 L_X(u)du=E\left(\left|X_1-X_2\right|\right)=E(X)-E(\min(X_1,X_2))=-2\operatorname{cov}\left(X,\left(1-F_X(X)\right)\right),$$

[15] See Shalit and Yitzhaki (1984) and the references therein

where X_1 and X_2 are two independent copies of *X*. GMD depends on the spread of the observations among themselves and not on the deviations from some central value. Consequently, this measure relates location with variability, two properties that Gini himself argue are distinct and do not depend on each other. While the Gini index, i.e. the ratio GMD/E(X), [16] has been used for the past 80 years as a measure of income inequality, the interest in GMD as a measure of risk in portfolio selection is relatively recent[17]. Alternatively to GMD we can consider the extended Gini's mean difference that takes into account the degree of risk aversion as reflected by the parameter *v*. As we can see from the following representation, this index can also be expressed as a function of the Lorenz curve:

$$\Gamma_X(v) = E(X) - v(v-1)\int_0^1 (1-u)^{v-2} L_X(u)du = -v\operatorname{cov}\left(X, \left(1-F_X(X)\right)^{v-1}\right) \quad . \qquad (21)$$

From this definition, it follows that the measures $\Gamma_X(v) - E(X) = -\Gamma(v+1)F_X^{(-(v+1))}(1)$ characterize the previous Gini FORS orderings. Thus, as a consequence of Remark 1 the extended GMD is a deviation measure associated with the expected bounded coherent risk measure $\Gamma_X(v) - E(X)$ for every *v*>1. Interest in the potential applications to portfolio theory of GMD and its extension has been fostered by Shalit and Yitzhaki (1984), who have explained the financial insights of these measures. Moreover, using Gini measures, we can consider the Gini tail measure for a given β: associated with a dilation order (see Fagiuoli *et al* (1999)) $\Gamma_{X,\beta}(1) = E(X) - \left(\int_0^\beta F_X^{-1}(u)du\right)\Big/\beta$, that can also be extended considering $v > 1$ and using the tail measure:

$$\Gamma_{X,\beta}(v) = E(X) - v\int_0^\beta (\beta-u)^{v-1} F_X^{-1}(u)du\Big/\beta^v = E(X) - v(v-1)\int_0^\beta (\beta-u)^{v-2} L_X(u)du\Big/\beta^v \qquad (22)$$

for some $\beta \in [0,1]$. Even in this case, $\Gamma_{X,\beta}(v) = E(X) - \Gamma(v+1)F_X^{(-(v+1))}(\beta)\big/\beta^v$ (for every *v*>1) is the deviation measure associated with the expected bounded coherent risk measure $\Gamma_{X,\beta}(v) - E(X)$. Moreover, if *X RSD Y* then *E*(*X*) = *E*(*Y*) and $X \underset{-2}{\geq} Y$. Thus, $-\Gamma(v+1)F_X^{(-(v+1))}(\beta)\big/\beta^v \leq \Gamma_{Y,\beta}(v) - E(Y)$ for

[16] In the income inequality literature, the Gini index is the area between the relative Lorenz curve and the 45° line expressing complete equality
[17] See Shalit and Yitzhaki (1984).

every $v \geq 1$ and for every $\beta \in [0,1]$, i.e. $\Gamma_{X,\beta}(v) \leq \Gamma_{Y,\beta}(v)$. This means that $\Gamma_{X,\beta}(v)$ is an uncertainty measure consistent with *R-S* order. Moreover, $E(X)-E(Y)=\Gamma(v+1)\int_0^{\beta}\left(1-\frac{u}{\beta}\right)^{v} d\left(F_X^{-1}(u)-F_Y^{-1}(u)\right)$ for any $\beta \in [0,1]$ if and only if $F_X = F_Y$ if and only if $\Gamma_{X,\beta}(v)=\Gamma_{Y,\beta}(v)$ for any $\beta \in [0,1]$, as remarked in the following:

Remark 2 *For every $v \geq 1$ and for every $\beta \in (0,1)$ the Gini tail measure $\Gamma_{X,\beta}(v) = E(X) - \Gamma(v+1) F_X^{(-(v+1))}(\beta)/\beta^{v}$ is consistent with R-S order. Moreover, if $\Gamma_{X,\beta}(v)=\Gamma_{Y,\beta}(v)$ for any $\beta \in [0,1]$, then $F_X = F_Y$.*

From Remark 2 we deduce that for any $v \geq 1$, we can introduce a new uncertainty FORS ordering induced by R-S ordering. That is, we say that *X* dominates *Y in the sense of v-Gini uncertainty ordering* if and only if $\Gamma_{X,\beta}(v) \leq \Gamma_{Y,\beta}(v)$ for any $\beta \in [0,1]$. These new uncertainty orderings generalize the dilation order that holds when *v*=1 (see Fagiuoli *et al* (1999)). Observe that although we can minimize a Gini tail measure $\Gamma_{X,\beta}(v)$ by solving an optimization problem with linear objective function, we cannot easily maximize $\Gamma_{X,\beta}(v)$ with the same linear optimization problem because the concentration curve appears with a negative sign.

Therefore, if investors want to optimize a portfolio of European options as suggested in Section 1, they could maximize the uncertainty with respect the *v*-Gini ordering by solving the following optimization problem for some values *m,* q_i; $i=1,2$ and $\beta = s/T$ (with an integer $s \leq T$):

$$\begin{aligned}
&\max_x \frac{1}{T}\sum_{t=1}^{T} x' z_{(t)} - \frac{v}{\beta^{v} T}\sum_{k=1}^{s-1}\left(\frac{s-i}{T}\right)^{v-1}\left(x'z\right)_{k:T} \text{ subject to} \\
&\frac{E\left(\left(x'R-E\left(x'R\right)\right)^{3}\right)}{\left(x'Q_R x\right)^{3/2}} \geq q_1;\ \frac{E\left(\left(x'R-E\left(x'R\right)\right)^{4}\right)}{\left(x'Q_R x\right)^{2}} \leq q_2 \\
&\frac{1}{T}\sum_{k=1}^{T} x' R_{(k)} \geq m\, x' Q_R x;\ \sum_{j=1}^{n} x_j = 1;\ \ x_j \geq 0;\ \ j=1,\dots,n;
\end{aligned} \tag{23}$$

where $\left(x'z\right)_{k:T}$ is the *k*-th of *T* ordered observations of the underlying log returns portfolio $x'z$. The portfolio optimization problems (19), (20), and (23) that propose to maximize the uncertainty of the underlying log returns are not computationally simple compared to those presented in Section 2. There exists a theoretical motivation to this fact. As a matter of fact, Bauerle and Müller (2006) have proved that measures which satisfy the convexity property (such as the functionals (16), (17), and (22)) are uncertainty measures. Thus, when we maximize the uncertainty on a compact set of choices, we could have more local optima. Instead, if we want to minimize the uncertainty, we can easily linearize all the functionals (16), (17), and (22) using arguments similar to those of Section 2**.**

3.2 Practical Portfolio Problems Consistent with Tracking Error Orderings

As shown by Stoyanov *et al*. (2008), there is a strong connection between probability metric theory and portfolio theory. In this subsection, we first recall some of the basic properties of probability distances that under some opportune assumption can be also used as uncertainty measures (concentration/dispersion measures), then we introduce the concept of FORS tracking error orderings and measures. Thus, we propose portfolio problems where a benchmark asset with return r_Y and n risky assets with returns $r=[r_1,...,r_n]'$ are traded. In particular, we examine portfolio problems consistent with some particular tracking error orderings using T i.i.d. observations of the vector of returns $r_{(k)}=[r_{1,k},...,r_{n,k}]'$, and of the benchmark return $r_{Y,k}$, $k=1,\dots,T$.

As observed in Section 1, often investors want to reduce a distance with respect to a given benchmark. Any probability functional μ is called a *probability distance* with parameter K if it is *positive* and satisfies the following additional properties:

1) *Identity* $f(X)=f(Y)\Leftrightarrow \mu(X,Y)=0$;

2) *Symmetry* $\mu(X,Y)=\mu(Y,X)$

3) *Triangular inequality* $\mu(X,Z)\le K\left(\mu(X,Y)+\mu(Y,Z)\right)$ for all admissible random variables *X, Y,* and *Z*

where *f*(*X*) identifies some characteristics of the random variable *X*. If the parameter *K* equals 1, we have a *probability metric*. We can always define the alternative finite distance $\mu_H(X,Y)=H\left(\mu(X,Y)\right)$, where

$H:[0,+\infty)\to[0,+\infty)$ is a non-decreasing positive continuous function such that $H(0)=0$ and $K_H=\sup_{t>0}\frac{H(2t)}{H(t)}<+\infty$.[18] Therefore, for any probability metric μ, μ_H is a probability distance with parameter K_H. In this case, we distinguish between primary, simple, and compound probability distances/metrics that depend on certain modifications of the identity property (see Rachev (1991)). Compound probability functionals identify the random variable almost surely i.e.: $\mu(X,Y)=0 \Leftrightarrow \Pr(X=Y)=1$. Simple probability functionals identify the distribution (i.e., $\mu(X,Y)=0 \Leftrightarrow F_X=F_Y$). Primary probability functionals determine only some random variable characteristics. Often we can associate a *distance ordering* (on the space of random variables Λ) to compound (or simple) distances μ defined between a random variable belonging to Λ and a fixed benchmark $Z\in\Lambda=V$. The ordering depends on the metric/distance definition and is related to a FORS functional $\rho:U\times B\to\mathbb{R}$ (where (B,M_B) is a measurable space) that serves to order distances between distributions or random distances as explained in the following:

Definition 1 *We say X is preferred to Y with respect to the μ-compound (simple) distance from Z (namely $X\succ_{\mu_Z}Y$) if and only if there exists a probability functional $\rho:\Lambda\times Z\times B\to\mathbb{R}$ dependent on μ such that for any $t\in B$ and $X,Y\in\Lambda$, $\rho_X(t)\le\rho_Y(t)$. In this case, the equality $\rho_X=\rho_Y$ implies a distributional equality $F_{g(X,Z)}=F_{g(Y,Z)}$ for compound distances μ and a distance equality $g(F_X,F_Z)=g(F_Y,F_Z)$ for simple distances μ (where $g(x,z)$ is a distance in $\mathbb{R}$). We call ρ_X (tail) tracking error measure (functional) associated with the μ-FORS tracking error ordering.*

Consequently, in passive tracking error strategies we should minimize the functional ρ_X associated with the μ-FORS tracking error ordering as done in optimization problem (2). This is generally different from active tracking error strategies where investors want to outperform the benchmark and

[18] See Rachev (1991) for further generalizations.

optimize a particular non-symmetric probability functional (see, among others, Stoyanov *et al.* (2008)). As a matter of fact, for active tracking error strategies we should also require that the optimal portfolio reward measure be greater than the reward measure of the benchmark.

In essence, probability metrics can be used as tracking error measures. In solving the portfolio problem with a probability distance, we intend to "approach" the benchmark and change the perspective for different types of probability distances. Hence, if the goal is only to control the uncertainty of an investor's portfolio or to limit its possible losses, mimicking the uncertainty or the losses of the benchmark can be done using a primary probability distance. When the objective for an investor's portfolio is to mimic entirely the benchmark, a simple or compound probability distance should be used. In addition to its role as measuring tracking error, a compound distance can be used as a measure of uncertainty**.** As a matter of fact, if we apply any compound distance $\mu(X,Y)$ to *X* and $Y = X_1$ that are i.i.d., then we get:

$$\mu(X, X_1) = 0 \Leftrightarrow \Pr(X = X_1) = 1 \Leftrightarrow X \text{ \textit{is a constant almost surely.}}$$

For this reason, we refer to $\mu(X, X_1) = \mu_I(X)$ as a *concentration measure* derived by the compound distance μ. Similarly, if we apply any compound distance $\mu(X,Y)$ to *X* and $Y = E(X)$ (either $Y = M(X)$, i.e. the median or a percentile of *X*, if the first moment is not finite), we get:

$$\mu(X, E(X)) = 0 \Leftrightarrow \Pr(X = E(X)) = 1 \Leftrightarrow X \text{ \textit{is a constant almost surely.}}$$

Hence, $\mu(X, E(X)) = \mu_{E(X)}(X)$ can be referred to as a *dispersion measure* derived by the compound distance μ. Let's consider the following examples of compound metrics, the associated concentration, dispersion measures, tracking error orderings, and associated practical portfolio problems.

***3.2.1 Examples of Probability Compound Metrics*:**

As observed above, for each probability compound metric we can always generate a probability compound distance $\mu_H(X,Y) = H(\mu(X,Y))$ with parameter K_H.

L^p ***-metrics***: For every $p \geq 0$ we recall the L^p-metrics: $\mu_p(X,Y) = E\left(|X - Y|^p\right)^{\min(1,1/p)}$; the associated concentration measures are $\mu_{I,p}(X)=E\left(\left|X-X_1\right|^p\right)^{\min(1,1/p)}$, where X_1 is an i.i.d. copy of *X*; and the associated dispersion measures are the central moments $\mu_{E(X),p}(X)=E\left(\left|X-E(X)\right|^p\right)^{\min(1,1/p)}$. The dispersion and concentration measures $\mu_{E(X),p}(X)$ and $\mu_{I,p}(X)$ are uncertainty measures consistent with $(p+1)$ R-S order for any $p\geq 1$. We can consider for L^p metrics the FORS tracking error measures

$$\rho_{X,p}(t)=\left(\mu_p(XI_{[|X-Z|\geq t]},ZI_{[|X-Z|\geq t]})\right)^{1/\min(1,1/p)}-t^p\Pr\left(\left|X-Z\right|\geq t\right)=E\left(\left(\left|X-Z\right|^p-t^p\right)_+\right)$$

for any $t\in[0,+\infty)$ associated with an μ_p FORS tracking error ordering. Besides $\rho_{X,p}=\rho_{Y,p}$ implies that $F_{|X-Z|}=F_{|Y-Z|}$. Thus, all investors who choose portfolios consistent with this μ_p FORS tracking error ordering should solve the following portfolio selection problem for some given *t>0*:

$$\begin{aligned}&\min_x \frac{1}{T}\sum_{k=1}^{T}u_k \text{ subject to}\\&\sum_{j=1}^{n}x_j=1;\ \ x_j\geq 0;\ j=1,...,n; v_k\geq x'r_{(k)}-r_{Y,k}\\&v_k\geq r_{Y,k}-x'r_{(k)};\ u_k\geq v_k^p-t^p; u_k\geq 0; k=1,...,T\end{aligned}\tag{24}$$

that is linear when *p* = 1 and convex when $p>1$. Alternatively, we could introduce as FORS tracking error functional associated with an μ_p FORS tracking error ordering the probability functional $\rho_X(q)=\left(\mu_q(X,Z)\right)^{1/\min(1,1/q)}=E\left(\left|X-Z\right|^q\right)$ for any $q\in(0,p]$ that still identifies the distribution $F_{|X-Z|}$. So investors who choose portfolios consistent with this μ_p FORS tracking error ordering should solve the following portfolio selection problem with linear constraints for some $q\in(0,p]$:

$$\begin{aligned}&\min_x \frac{1}{T}\sum_{k=1}^{T}v_k^q \text{ subject to}\\&\sum_{j=1}^{n}x_j=1;\ \ x_j\geq 0; j=1,...,n; v_k\geq x'r_{(k)}-r_{Y,k};\\&v_k\geq r_{Y,k}-x'r_{(k)};\ k=1,...,T.\end{aligned}\tag{25}$$

That is, a convex programming problem when $p>1$ and we use $q\in[1,p]$.

Ky Fan metrics: $k_1(X,Y)=\inf\{\varepsilon>0/\Pr(|X-Y|>\varepsilon)<\varepsilon\}$ and $k_2(X,Y)=E\left(\frac{|X-Y|}{1+|X-Y|}\right)$, the respective concentration measures are $k_{1,I}(X)=\inf\{\varepsilon>0/\Pr(|X-X_1|>\varepsilon)<\varepsilon\}$, $k_{2,I}(X)=E\left(\frac{|X-X_1|}{1+|X-X_1|}\right)$, while the associated dispersion measures are $k_{1,E(X)}(X)=\inf\{\varepsilon>0/\Pr(|X-E(X)|>\varepsilon)<\varepsilon\}$, $k_{2,E(X)}(X)=E\left(\frac{|X-E(X)|}{1+|X-E(X)|}\right)$.

For Ky Fan metrics we consider the FORS tracking error measure $\rho_{X,2}(t)=k_2(XI_{[|X-Z|\geq t]},ZI_{[|X-Z|\geq t]})$ associated with the k_2-FORS tracking error ordering, where $\rho_{X,2}=\rho_{Y,2}$ implies $F_{|X-Z|}=F_{|Y-Z|}$. Therefore, we get choices consistent with k_1 FORS tracking error metric by solving the optimization problem:

$$\begin{aligned}&\min_x\ u \qquad \text{subject to}\\ &\sum_{j=1}^n x_j=1;\quad x_j\geq 0; u\geq 0;\\ &\frac{1}{T}\sum_{k=1}^T I_{[|x'r_{(k)}-r_{Y,k}|>u]}<u.\end{aligned} \tag{26}$$

On the other hand, investors who choose portfolios consistent with this k_2 FORS tracking error ordering should minimize the consistent estimator $\hat{\rho}_{X,2}(t)=\frac{1}{T}\sum_{k=1}^T\frac{|x'r_{(k)}-r_{Y,k}|I_{[|x'r_{(k)}-r_{Y,k}|\geq t]}}{1+|x'r_{(k)}-r_{Y,k}|}$ for some *t>0.*

Generally, when we consider the compound metric/distance as dispersion/concentration measure $\mu(X,f(X))$ (where *f*(*X*) is either a functional of *X* or an independent copy of *X*), we should obtain a tracking error measure between *X* and *Z* using $\mu(X-Z,f(X-Z))$. In particular, some of these tracking error type measures (i.e., $\mu(X-Z,f(X-Z))$) have been used in the portfolio literature by, for example, Stoyanov *et al.* (2008) .

Moreover, even simple probability distances can be used as dispersion measures and tracking error measures, but, generally not as concentration measures. As a matter of fact, when we apply any simple distance $\mu(X,Y)$ to *X* and $Y=E(X)$ (either $Y=M(X)$, i.e., median or a percentile of *X*, if the first moment is not finite), we get:

$$\mu(X,E(X))=0 \Leftrightarrow F_X = F_{E(X)} \Leftrightarrow X \ \textit{is a constant almost surely.}$$

Thus, we refer to $\mu(X,E(X))=\mu_{E(X)}(X)$ as a *dispersion measure* derived by the simple distance μ. As for compound metrics, we can generate a simple probability distance $\mu_H(X,Y)=H\left(\mu(X,Y)\right)$ with parameter K_H for any simple probability metric $\mu(X,Y)$. Let's consider the following examples of simple metrics, the associated dispersion measures, and FORS tracking error orderings.

***3.2.2 Examples of Simple Probability Metrics*:**

FORS tracking error simple metrics and downside risk measures: Consider a frictionless economy where a benchmark asset with return r_Y and $n \geq 2$ risky assets with returns $r=[r_1,...,r_n]'$ are traded. Let $\rho_X:[a,b]\to R$ be a FORS measure associated with a FORS risk ordering defined over any admissible portfolio of returns $X=x'r$ and over the return $Y=r_Y$. Then we define for any positive p the FORS tracking error metric:

$$\rho_{x'r,r_Y}(p)=\left(\int_a^b\left|\rho_{x'r}(\lambda)-\rho_{r_Y}(\lambda)\right|^p d\lambda\right)^{\min(1/p,1)}.$$

Similarly, we describe the associated dispersion measures whose definition depends on the definition of the functional ρ_X. For any of these FORS tracking error metrics, we consider the FORS tracking error orderings with the associated (tail) FORS tracking error measures:

$$\rho_{x'r,r_Y}(p,t)=\left(\int_a^b\left|\rho_{x'rI_{[x'r\leq t]}}(\lambda)-\rho_{r_YI_{[r_Y\leq t]}}(\lambda)\right|^p d\lambda\right)^{\min(1/p,1)} \quad \forall t\in[a,b],$$

where $\rho_{x'r,r_Y}(p,.)=\rho_{w'r,r_Y}(p,.)$ if and only if $\left|\rho_{x'r}-\rho_{r_Y}\right|=\left|\rho_{w'r}-\rho_{r_Y}\right|$. In addition, $x'r \underset{\succ}{FORS}\, r_Y$, if and only if $\int_a^b\left(\rho_{x'r}(\lambda)-\rho_{r_Y}(\lambda)\right)_+^u d\lambda=0$. Thus, when investors want to outperform the benchmark r_Y, they minimize the following non-symmetric measures

$$\rho_{x'r,r_Y}^{dsr}(u)=\left(\int_a^b\left(\rho_{x'r}(\lambda)-\rho_{r_Y}(\lambda)\right)_+^u d\lambda\right)^{\min(1/u,1)},$$

that we call *portfolio FORS downside risk measures*. Observe that $\rho^{dsr}_{x'r-E(x'r),r_Y-E(r_Y)}(u)$ is a relative deviation metric in the sense of Stoyanov *et al.* (2008). Generally, the benchmark r_Y is not dominated in the sense of $\underset{\succ}{FORS}$, otherwise the minimization of the portfolio FORS downside risk leads to infinite solutions. When the benchmark r_Y is not dominated, we can consider the FORS downside tracking error ordering with the associated functional

$$\rho^{dsr}_{x'r,r_Y}(p,t)=\left(\int_a^b\left(\rho_{x'rI_{[x'r\le t]}}(\lambda)-\rho_{r_YI_{[r_Y\le t]}}(\lambda)\right)_+^p d\lambda\right)^{\min(1/p,1)}, \ \forall t\in[a,b],$$

where $\rho^{dsr}_{x'r,r_Y}(p,.)=\rho^{dsr}_{w'r,r_Y}(p,.)$ if and only if $\left(\rho_{x'r}-\rho_{r_Y}\right)_+=\left(\rho_{w'r}-\rho_{r_Y}\right)_+$. In this case, when $\rho^{dsr}_{x'r,r_Y}(p,t)\le\rho^{dsr}_{w'r,r_Y}(p,t)$ for any *t* belonging to [*a,b*], we say that portfolio *x'r represents a higher out performance* than *w'r* in the sense of $\underset{\succ}{FORS}$ ordering and with respect to the common benchmark r_Y. Clearly, when we optimize a FORS downside risk measure we generally require that the optimal portfolio presents a higher reward measure than the benchmark. Typical examples of FORS tracking error measures and FORS downside risk measures are those based on Gini type metrics and Generalized Zolotarev ones which are discussed next.

Generalized Zolotarev metric: For every $q\ge 0$, the generalized Zolotarev metric among variates *X, Y* with support on the interval [*a,b*] is given by $GZM_{q,\alpha}(X,Y)=\left(\int_a^b\left|F_X^{(\alpha)}(t)-F_Y^{(\alpha)}(t)\right|^q dt\right)^{\min(1/q,1)}$ and the associated dispersion measure is $GZM_{q,\alpha}(X,E(X))=\left(\int_a^{E(X)}\left|F_X^{(\alpha)}(t)\right|^q dt+\int_{E(X)}^b\left|F_X^{(\alpha)}(t)-\frac{(t-E(X))^{\alpha-1}}{\Gamma(\alpha)}\right|^q dt\right)^{\min(1/q,1)}$.

This metric has been introduced by Zolotarev for q=1 and extended by Rachev (1991) for $q\ne 1$. Associated to a $GZM_{p,\alpha}$ FORS tracking error ordering we can consider the FORS tracking error measure $\rho_X(t)=GZM_{p,\alpha}(XI_{[X\le t]},ZI_{[Z\le t]})^{1/\min(1/p,1)}$ where $\rho_X=\rho_Y$ implies that $\left|F_X^{(\alpha)}-F_Z^{(\alpha)}\right|=\left|F_Y^{(\alpha)}-F_Z^{(\alpha)}\right|$. Then, for any $t\ge c=\min_{1\le k\le T} r_{Y,k}$, a consistent estimator of $GZM_{p,\alpha}(x'rI_{[x'r\le t]},r_YI_{[r_Y\le t]})^{1/\min(1/p,1)}$ for *M* large is the functional $\tilde{G}^{(\alpha,p)}_{x'r,r_Y}(t)=\left(\frac{t-c}{M}\sum_{i=0}^{M}\left|\frac{1}{T\Gamma(\alpha)}\sum_{k=1}^{T}\left(c+i\frac{t-c}{M}-x'r_{(k)}\right)^{\alpha-1}I_{\left[c+i\frac{t-c}{M}>x'r_{(k)}\right]}-\left(c+i\frac{t-c}{M}-r_{Y,k}\right)^{\alpha-1}I_{\left[c+i\frac{t-c}{M}>r_{Y,k}\right]}\right|^{p}\right)$,

since $\tilde{G}^{(\alpha,p)}_{x'r,r_Y}(t)\xrightarrow{M,T\to\infty}GZM_{p,\alpha}(x'rI_{[x'r\le t]},r_YI_{[r_Y\le t]})^{1/\min(1/p,1)}$. Thus all investors that choose portfolios consistent with this $GZM_{p,\alpha}$ FORS tracking error ordering should solve the following portfolio selection problem for some given $t\ge c=\min_{1\le k\le T} r_{Y,k}$:

$$\begin{gathered}\min_x \ \tilde{G}^{(\alpha,p)}_{x'r,r_Y}(t) \ \text{ subject to} \\ \sum_{j=1}^{n}x_j=1;\ \ x_j\ge 0;\ \ j=1,...,n.\end{gathered} \tag{27}$$

Gini's Index of Dissimilarity and Extensions: To measure the degree of difference between two random variables, Gini introduced the *index of dissimilarity*. The index properly measures the distance between two variates and has been intensively used in mass transportation problems. We present here some of the many representations of Gini's index of dissimilarity:

$$G_{X,Y}(1)=\int_{-\infty}^{+\infty}\left|F_X(x)-F_Y(x)\right|dx=\int_0^1\left|F_X^{-1}(u)-F_Y^{-1}(u)\right|du=\inf\left\{E_F\left(\left|\tilde{X}-\tilde{Y}\right|\right)/F\in\Im(F_X,F_Y)\right\}=E_{\tilde{F}}\left(\left|\tilde{X}-\tilde{Y}\right|\right)$$

where $\tilde{X}=F_X^{-1}(U)$, $\tilde{Y}=F_Y^{-1}(U)$, U is uniformly distributed (0,1), and $\tilde{F}(x,y)=\min\left(F_X(x),F_Y(y)\right)$ is the Hoeffding-Frechet bound of the class of all bivariate distribution functions $\Im(F_X,F_Y)$ with marginals F_X and F_Y (see Rachev (1991)). In portfolio theory, this risk measure evolves with respect to the chosen benchmark. For example, when the mean E(*X*) is used as benchmark *Y*, the index of dissimilarity is the mean absolute deviation of *X,* a dispersion measure consistent with the R-S stochastic order.

A logical expansion of Gini's index of dissimilarity is obtained by the FORS tracking error metrics $G_{X,Z,\alpha}(q)=\left(\int_0^1\left|F_X^{(-\alpha)}(p)-F_Z^{(-\alpha)}(p)\right|^q dp\right)^{\min(1/q,1)}$, and the associated downside risk tracking error measures $G^{dsr}_{X,Z,\alpha}(q)=\left(\int_0^1\left(F_Z^{(-\alpha)}(p)-F_X^{(-\alpha)}(p)\right)_+^q dp\right)^{\min(1/q,1)}$. As in Section 2, we can simplify for these measures the optimization by considering some linearizations when $\alpha\ge 2$. Thus, for example, if we want to minimize the tail FORS tracking error measure $G^{dsr}_{x'r,r_Y,2}(q,p)$ for some $p = s/T$ we should solve the following simplified portfolio selection problem:

$$\min_{x;a_i;i=1,\dots,s} \frac{1}{T}\sum_{t=1}^{s} z_t^q \text{ subject to}$$

$$z_t \geq \sum_{i=1}^{t}\left(\frac{i}{T}a_i + \frac{1}{T}\sum_{k=1}^{T} v_{k,i} + \frac{1}{T}\sum_{k=1}^{i}\left(r_Y\right)_{k:T}\right); z_t \geq 0 \text{ for } t = 1,\dots,T \tag{28}$$

$$v_{k,i} \geq 0;\ v_{k,i} \geq -x'r_{(k)} - a_i,\ \frac{1}{\mathrm{T}}\sum_{t=1}^{T} x'r_{(t)} \geq E(r_Y) \quad k = 1,\dots,T; i = 1,\dots,s$$

where $r_t = [r_{1,t},\dots,r_{n,t}]'$ is the vector of returns at time *t*, $\left(r_Y\right)_{k:T}$ is the *k*-th of *T* ordered observations of r_Y, and the optimal values -a_i are the (i/T)*-th* percentiles of the portfolio *x'r.*

4. Concluding remarks

The first contribution of this paper consisted in classifying three possible portfolio problems with respect to the type of ordering required, that could be either a risk ordering, an uncertainty ordering, or a distance ordering. By doing so, we introduced new risk and behavioral risk orderings, new Gini uncertainty orderings, and new distance orderings that are used to better classify investors choices. Furthermore, we proposed new coherent risk measures, uncertainty measures, and tracking error measures based on probability functionals consistent with some stochastic orderings. The second contribution of the paper lies in the computational applicability of the portfolio problems arising from optimizing a risk measure, an uncertainty measure, or a probability distance. Thus, for each opportune orderings, we proposed several practical portfolio optimization problems that could be solved even for large portfolios when we consider choices consistent with risk orderings.

Several new perspectives and problems arise from this analysis. Since we can better specify the problem of portfolio optimization by taking into account the attitude of investors toward risk, we have to consider the ideal characteristics of the associated statistics and their asymptotic behavior. By using the theory of probability metrics, we can explain and argue why a given metric must be used for a particular optimization problem.

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