Distortion Risk Measures in Portfolio Optimization

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Abstract

Distortion risk measures are perspective risk measures because they allow an asset manager to reflect a client's attitude toward risk by choosing the appropriate distortion function. In this paper, the idea of asymmetry was applied to the standard construction of distortion risk measures. The new asymmetric distortion risk measures are derived based on the quadratic distortion function with different risk-averse parameters.

Key words and phrases: risk measures, distortion risk measures, asymmetry, portfolio optimization

1 Introduction

The selection of the appropriate portfolio risk measures continues to be a topic of heated discussion and intensive investigations in investment management, as all the proposed risk measures have drawbacks and limited applications. The major focus of researchers[1] has been on the "right" or "ideal" risk measure to be applied in portfolio selection. The principal complexity, however, is that the concept of risk is highly subjective, because every market player has its own perception of risk. Consequently, Balzer (2001) concludes, there is "no single universally acceptable risk measure." He suggests the following features that an investment risk measure should satisfy: relativity of risk, multidimensionality of risk, asymmetry of risk, and non-linearity.

Rachev et al. (2005) summarize the desirable properties of an "ideal" risk measure, capturing fully the preferences of investors. These properties relate to investment diversification, computational complexity, multi-parameter dependence, asymmetry, non-linearity, and incompleteness. However, every risk measure proposed in the literature possesses only some of these properties. Consequently, proposed risk measures are insufficient and, based on this, Rachev et al. (2005, p. 4) conclude that an ideal measure does not exist. However, they note that "it is reasonable to search for risk measures which are ideal for the particular problem under investigation."

Historically, the most commonly used risk measure is the standard deviation (variance) of a portfolio's return. In spite of its computation simplicity, variance is not a satisfactory measure due to its symmetry property and inability to consider the risk of low probability events.

A risk measure that has received greater acceptance in practice is value at risk (VaR). Unfortunately, because VaR fails to satisfy the sub-additivity property and ignores the potential loss beyond the confidence level, researchers and practitioners[2] have come to realize its limitations, limiting its use for reporting purposes when regulators require it or when a simple to interpret number is required by clients.

[1]See e.g. Goovaerts et al. (1984), Artzner et al. (1999), Kaas et al. (2001), Goovaerts et al. (2003), Zhang and Rachev (2004), Denuit et al. (2006), Rachev et al. (2005).

[2]See e.g. Arztner et al. (1999), Szegö (2004), Zhang and Rachev (2004).

A major step in the formulation of a systematic approach towards risk measures was taken by Artzner et al. (1999). They introduced the notion of “coherent” risk measures. It turns out that VaR is not a coherent risk measure. In contrast, a commonly used risk measure in recent years, conditional value at risk (CVaR), developed by Rockafellar and Uryasev (2002), is, in fact, a coherent risk measure. The most general theoretical result about coherent measures is the class of distortion risk measures, introduced by Denneberg (1990) and Wang et al. (1997).

Distortion risk measures were obtained by the simultaneous use of two approaches[3] to define the particular class of risk measures: axiomatic definition and the definition from the economic theory of choice under uncertainty. Due to the second approach, distortion risk measures have their roots in the dual utility theory of Yaari (1987). Using the expected utility's set of axioms with a modified independence axiom, Yaari (1987) developed the distortion utility theory. He has shown that there must exist a “distortion function” such that a prospect is valued at its distorted expectation. Instead of using the tail probabilities in order to quantify risk, the decision maker uses the distorted tail probabilities. For the axiomatic definition, Wang et al. (1997) postulated the axioms to characterize the price of insurance risk. These axioms include the following: law invariance, monotonicity, co-monotonic additivity, and continuity. They also proved that risk measures hold such properties if and only if they have the Choquet integral representation with respect to a distorted probability.

Distortion risk measures were originally applied to a wide variety of insurance problems such as the determination of insurance premiums, capital requirement, and capital allocation. Because insurance and investment risks are closely related, the investment community started to apply distortion risk measures in the context of the asset allocation problem[4]. Wang (2004) has applied the distortion risk measure to price catastrophe bonds and Fabozzi and Tunaru (2008) to price real estate derivatives.

In the application of portfolio selection, distortion risk measures with the concave distortion function reveal the desired properties, such as law-invariance, sub-additivity, and consistency with second-order stochastic dominance. Law-invariance is the prerequisite for the ability to quantify risk of a portfolio from historical data. Sub-additivity secures the diversification effect. In general, the motif of constructing a portfolio is to reduce overall investment risk through diversification as set forth by Markowitz (1952), that is, investing in different asset classes and in securities of many issuers. Diversification ensures the avoidance of extreme poor portfolio performance caused by the underperformance of a single security or an industry. The consistency with second-order stochastic dominance provides the link between the construction of risk measures and the decision theory under uncertainty.

In this paper, we propose new distortion risk measures, adding the asymmetric property to the already existing properties of concave distortion risk measures. We do so by extending

[3]The combination of two approaches – as demonstrated by Föllmer and Schied (2002a), Tsanakas and Desli (2003), and Denuit et al. (2006) – add better perception of the inherent properties of risk measures.

[4]See, for example, van der Hoek and Sherris (2001), Gourieroux and Liu (2006), Hamada et al. (2006), and Balbas et al. (2007).

the Choquet integral construction using quadratic and power distortion functions with different concave parameters in order to better capture the risk perception of investors.

The paper is organized as follows. Section 2 starts with the general definition of risk and provides various classifications of risk measures that have appeared in the literature. Sections 3 and 4 provide a discussion of the distortion risk measures and their properties. In Section 5, we give examples of distortion functions and show how distortion risk measures are related to VaR and CVaR. We propose the new distortion risk measures with asymmetric property in Section 6. Section 7 summarizes our paper. The appendix contains the properties of risk measures and we make reference to them by property number in the body of the paper.

2 Classes of risk measures

In general, a risk measure, $\rho\colon \mathcal{X} \to \mathbb{R}$, is a functional that assigns a numerical value to a random variable representing an uncertain payoff. $\mathcal{X}$ is defined on $L^\infty(\Omega, \mathcal{F}, P)$[5], the space of all essentially bounded random variables defined on the general probability space $(\Omega, \mathcal{F}, P)$. Not every functional corresponds to the intuitive notion of risk. One of the main characteristics of such a function is that a higher uncertain return should conform to a higher functional value.

Goovaerts et al. (1984) presented the pioneering work of the axiomatic approach to risk measures in actuarial science, where risk measures were analyzed within the framework of premium principles. Artzner et al. (1999) extended the use of this axiomatic approach in the financial literature. The axiomatic definition of how risk is measured includes the setting of the assorted properties (axioms) on a random variable and then the determination of the mathematical functional fitting to the set of axioms.

2.1 Pederson and Satchell's class of risk measures

Pederson and Satchell (1998) define risk as a deviation from a location measure. They provided four desirable properties of a "good financial risk measure", such as nonnegativity, positive homogeneity, sub-additivity, and translation invariance[6]. Pedersen and Satchell also presented in their work the full characterization of the appropriate risk measures according to their system of axioms.

[5] It could be efficient to use unbounded random variables for modeling risks, as far as financial risk has no limits. The implications of risk measure' properties can be different on certain finite and on non-atomic probability spaces. Check Bäuerle and Müller (2006) and Inoue (2003) for further results on the extension from L^∞ to L^1 probability space.

[6] *Property 8*, *property 2*, *property 3.1*, and *property 6.3*, respectively.

2.2 Coherent risk measures

The idea of coherent risk measures was introduced by Artzner et al. (1999). Coherent risk measures are those measures which are translation invariant, monotonous, sub-additive, and positively homogeneous[7]. Coherent measures have the following general form:

$$\rho(X) = \sup_{Q \in \mathcal{Q}} E_Q[-X],$$

where $\mathcal{Q}$ is some class of probability measures on Ω.

Four criteria proposed by Artzner et al. (1999) provide rules for selecting and evaluating risk measures. However, one should be aware that not all risk measures satisfying the four proposed axioms are reasonable to use under certain practical situations. Wang (2002) argued that "a risk measure should go beyond coherence" in order to utilize useful information in a large part of a loss distribution. Dhaene et al. (2003), observing "best practice" rules in insurance, concluded that coherent risk measures "lead to problems".

2.3 Convex risk measures

Convex risk measures (also called weakly coherent risk measures) were studied by Föllmer and Schied (2002a, 2002b) and Frittelli and Rosazza Gianin (2005). Convex risk measures are a generalization of coherent risk measures obtained by relaxation of the positive homogeneity assumption (*property 2*) together with the sub-additivity condition (*property 3.1*) and require the weaker property of convexity (*property 4*). Any convex risk measure takes into account a nonlinear increase of the risk with the size of the position and has the following structure:

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \left(E_Q[-X] - \alpha(Q) \right),$$

where α is a penalty function defined on probability measures on Ω.

Following Frittelli and Rosazza Gianin (2005), a functional $\rho\colon \mathcal{X} \to \mathbb{R}$ is a convex risk measure if it suffices convexity (*property 4*), lower semi-continuity (*property 9.5*), and normalization ($\rho(0) = 0$) conditions. Bäuerle and Müller (2006) proposed replacing the convexity axiom by the weaker but more intuitive property of consistency with respect to convex order (*property 7.6*).

2.4 Law invariant coherent risk measures

Following the notation of Kusuoka (2001), law invariant coherent risk measures have the form:

$$\rho_\alpha(X) \triangleq \frac{1}{\alpha} \int_{1-\alpha}^{1} Z_{-X}(x)\, dx,$$

[7] *Property 6.3*, *property 5*, *property 3.1*, and *property 2*, respectively.

$Z\colon [0,1) \to \mathbb{R}$ is non-decreasing and right continuous. This class of risk measures satisfies the lower semi-continuity property (*property 9.5*) for all $X \in L^\infty$, $0 \le \alpha \le 1$. The class of insurance prices characterized by Wang et al. (1997) is an example of law invariant coherent risk measures.

2.5 Spectral risk measures

Spectral measures of risk[8] can be defined by adding two axioms to the set of coherency axioms: law invariance (*property 1*) and comonotonic additivity (*property 3.2*). Spectral risk measures consist of a weighted average of the quantiles of the returns distribution using a non-increasing weight function[9] referred to as a spectrum and denoted by ϕ. It is defined as follows:

$$M_\phi(X) = -\int_0^1 \phi(x) F_{\overline{X}}(x)\, dx,$$

where ϕ is a non-negative, non-increasing, right-continuous integrable function defined on $[0,1]$ and such that $\int_0^1 \phi(x)\, dx = 1$. Assumptions made on ϕ determine the coherency of spectral risk measures. If any of these assumptions is relaxed, the measure is no longer coherent. Spectral risk measures possess positive homogeneity (*property 2*), translation invariance (*property 6.3*), monotonicity (*property 5*), sub-additivity (*property 3.1*), law invariance (*property 1*), comonotonic additivity (*property 3.2*), consistency with second-order stochastic dominance (SSD) (*property 7.4*), and expected utility theory.

2.6 Deviation measures

Rockafeller et al. (2002)[10] defined deviation measures as positive, sub-additive, positively homogeneous, Gaivoronsky-Pflug (G-P) translation invariant[11] risk measures. Deviation measures are normally used by the totally risk-averse investors.

2.7 Expectation-bounded risk measures

Rockafeller et al. (2002) proposed expectation-bounded risk measures, imposing the conditions of sub-additivity, positive homogeneity, translation invariance and additional property of expectation-boundedness[12]. There exists a corresponding one-to-one relationship between deviation measures and expectation-bounded risk measures. One can derive expectation-bounded coherent risk measures if additionally monotonicity (*property 5*) is satisfied.

[8] See Kusuoka (2001), Acerbi (2002), Adam et al. (2007).

[9] ϕ can be observed as a weight function reflecting an investor's subjective risk aversion.

[10] See also Rockafeller et al. (2003, 2006).

[11] *Property 8*, *property 3.1*, *property 2*, and *property 6.2*, respectively.

[12] *Property 3.1*, *property 2*, *property 6.3*, and *property 10*, respectively.

2.8 Reward measures

De Giorgi (2005) introduced the first axiomatic definition for reward measures and provided their characterization. According to de Giorgi, such measures should satisfy the following conditions: additivity, positive homogeneity, isotonicity with respect to SSD, and risk-free condition[13].

2.9 Parametric classes of risk measures

Stone (1973) defined a general three-parameter class of risk measures, which has the form

$$R[c,k,A] = \left(\int_A^{-\infty} |y-c|^k f(y)\, dy \right)^{1/k},$$

where $A, c \in \mathbb{R}$ and $k > 0$. Stone's class of risk measures includes several commonly used measures of risk and dispersion, such as the standard deviation, the semi-standard deviation, and the mean absolute deviation.

Pedersen and Satchell (1998) generalized Stone's class of risk measures and introduced the five-parameter class of risk measures:

$$R[A,c,\alpha,\theta,w(\cdot)] = \left[\int_A^{-\infty} |y-c|^\alpha w\left[F(y)\right] f(y) dy \right]^\theta$$

for some bounded function $w(\cdot)$, $A, c \in \mathbb{R}$, $\alpha > 0$, $\theta > 0$. This class of risk measures also include the lower partial moments as an extention of the Stone class. Ebert (2005) argues that because of the confusing number of parameters presented by Pedersen and Satchell, "it seems to be impossible to comprehend their meaning and their interaction".

2.10 Quantile-based risk measures

Quantile-based risk measures include value at risk, expected shortfall, tail conditional expectation, and worst conditional expectation. We describe each measure below.

Value at risk (VaR) specifies how much one can lose with a given probability (confidence level). Its formal definition is

$$VaR^\alpha(X) = -x^{(\alpha)} = q_{1-\alpha}(-X).$$

VaR has the following properties: monotonicity (*property 5*), positive homogeneity (*property 2*), translation invariance (*property 6.3*), law invariance (*property 1*), comonotonic additivity (*property 3.2*). VaR possesses the sub-additivity attribute (*property 3.1*) for joint-elliptically distributed risks (see Embrechts et al. (2002)), but this assumption is rare in practice.

[13] *Property 3.2*, *property 2*, *property 7.4*, and *property 13.2*, respectively.

Despite its simplicity and wide applicability, VaR is controversial. A common criticism among academics is that VaR is not sub-additive, hence not coherent and that VaR calculations lead to substantial estimation errors (see e.g. Arztner et al. (1999), Szegö (2004), and Zhang and Rachev (2004)). A risk manager should be aware of its limitations and use it properly.

Expected shortfall (ES), also known as tail (or conditional) VaR (see Rockafellar et al. (2002)), corresponds to the average of all VaR^{α}s above the threshold α:

$$ES^{\alpha}(X) = \frac{1}{1-\alpha}\int_{\alpha}^{1} VaR_u(X)\, du, \quad \alpha \in (0,1).$$

ES was proposed in order to overcome some of the theoretical weaknesses of VaR. ES has the following properties: law invariance (*property 1*), translation invariance (*property 6.3*), comonotonic additive (*property 3.2*), continuity (*property 9*), monotonicity (*property 5*), sub-additivy (*property 3.1*). ES, being coherent[14], was proposed by Artzner et al. (1999) as a "good" risk measure.

Tail conditional expectation (TCE) was proposed by Artzner et al. (1999) in the following form:

$$TCE^{\alpha}(X) = -E\{X|X \leq x^{(\alpha)}\},$$

TCE^{α} does not possess the sub-additivity property for general distributions; it is coherent only for continuous distributions.

Worst conditional expectation (WCE) is defined as

$$WCE_{\alpha}(X) = -\inf\{E[X|A]\colon A \in \mathcal{F}, P(A) > \alpha\}.$$

$WCE_{\alpha}(X)$ is not law-invariant, so it cannot be estimated solely from data. Using such measures can lead to different risk values for two portfolios with identical loss distributions.

Comparing ES, TCE, and WCE, one finds that

$$TCE^{\alpha}(X) \leq WCE_{\alpha}(X) \leq ES_{\alpha}(X).$$

ES has the maximum value among TCE and WCE when the underlying probability law varies. If the distribution of X is continuous, then

$$TCE^{\alpha}(X) = WCE_{\alpha}(X) = ES_{\alpha}(X).$$

2.11 Drawdown measures

Drawdown measures are intuitive measures. A psychological issue in handling risk is the tendency of people to compare the current situation with the very best one from the past. Drawdowns measure the difference between two observable quantities - local maximum and local minimum of the portfolio wealth. Cheklov et al. (2003) defined the drawdown

[14] See Acerbi and Tasche (2004).

function as the difference between the maximum of the total portfolio return up to time t and the portfolio value at t.

Drawdown measures are close to the notion of deviation measure. Examples of drawdown measures constitute absolute drawdown (AD), maximum drawdown (MDD), average drawdown (AvDD), drawdown at risk (DaR), and conditional drawdown at risk (CDaR). In spite of their computational simplicity, drawdown measures cannot describe the real situation on the market, and therefore, should be used in combination with other measures.

3 Distortion risk measures

A *distortion risk measure* can be defined as the distorted expectation of any non-negative loss random variable X. It is accomplished by using a "dual utility" or the distortion function g[15] as follows:

$$\rho_g(X) = \int_0^\infty g(1 - F_X(x))\, dx = \int_0^1 F_X^{-1}(x)(1-q)\, dg(q), \tag{1}$$

where $g\colon [0,1] \to [0,1]$ is a continuous increasing function with $g(0) = 0$ and $g(1) = 1$; $F_X(x)$ denotes the cumulative distribution function of X, while $g(F_X(x))$ is referred to as a distorted distribution function.

For the gain/loss-distributions, when the loss random variable can take any real number, the distortion risk measure is obtained as follows:

$$\rho_g(X) = \int_0^1 F_X^{-1}(x) dH(x) = -\int_{-\infty}^0 H(F_X(x))dx + \int_0^\infty [1 - H(F_X(x))]dx,$$

where $H(u) = 1 - g(1-u)$. A similar expression holds if we use the survival function $S_X(x) = 1 - F_X(x) = P(X > x)$ instead of the distribution function,

$$\rho_g(X) = -\int_{-\infty}^0 [1 - g(S_X(x))]dx + \int_0^\infty g(S_X(x))dx.$$

Van der Hoek and Sherris (2001) developed a more general class of distortion risk measures, depending on the choice of parameters α, g, and h. It has the following form:

$$H_{\alpha,g,h}(X) = \alpha + H_h\left((X-\alpha)^+\right) - H_g\left((\alpha - X)^+\right),$$

where $\alpha^+ = \max[0, \alpha]$. When $\alpha = 0$ and $h(x) = 1 - g(1-x)$, then we again obtain the Choquet integral representation.

[15]Consider the set function $g\colon \mathcal{F} \to [0,\infty)$, defined on the σ-algebra $\mathcal{F}$, such that $g(\emptyset) = 0$ and $A \subseteq B \Rightarrow g(P[A]) \le g(P[B])$, for $A, B \in \mathcal{F}$. Such a function g is called a distortion function, and $P[A]$, $P[B]$ – distorted probabilitites.

4 Properties of distortion risk measures

The properties of the distortion risk measures correspond to the following standard results about the Choquet integral (see Denneberg (1994)):

1. If $X \geq 0$, then $\rho_g(X) \geq 0$, monotonicity

2. $\rho_g(\lambda X) = \lambda \rho_g(X)$, for all $\lambda \geq 0$, positive homogeneity

3. $\rho_g(X + c) = \rho_g(X) + c$, for all $c \in \mathbb{R}$, translation invariance[16]

4. $\rho_g(-X) = -\rho_{\tilde{g}}(X)$, where $\tilde{g}(x) = 1 - g(1 - x)$[17]

5. If a random variable X_n has a finite number of values (i.e., $X_n \xrightarrow{\text{w}} X$) and $\rho_g(X)$ exists, then $\rho_g(X_n) \rightarrow \rho_g(X)$. This property implies that it is enough to prove the statement for the discrete random variables, and then carry over the result to the general continuous case.

6. If X and Y are comonotonic risks, taking positive and negative values, then

$$\rho_g(X + Y) = \rho_g(X) + \rho_g(Y)$$

In literature, this property is called *comonotonic additivity*.

[16]The proof is as follows:

$$\begin{aligned}\rho_g(X + c) &= \int_0^{-\infty} [g(S_{X+c}(x) - 1] \, dx + \int_0^c g(S_{X+c}(x)) \, dx + \int_c^\infty g(S_{X+c}(x)) \, dx \\ &= \int_{-\infty}^0 [g(S_X(x - c)) - 1] \, dx + \int_0^c g(S_X(x - c)) \, dx + \int_c^\infty g(S_X(x - c)) \, dx.\end{aligned}$$

By replacing $x = c + u$, we get

$$\begin{aligned}\rho_g(X + c) &= \int_{-\infty}^{-c} [g(S_X(u)) - 1] \, du + \int_{-c}^0 g(S_X(u)) \, du + \int_0^\infty g(S_X(u)) \, du \\ &= \int_{-\infty}^0 [g(S_X(u)) - 1] \, du + \int_0^\infty g(S_X(u)) \, du + \int_{-c}^0 \, du \\ &= \rho_g(X) + c.\end{aligned}$$

[17]The proof is as follows:

$$\begin{aligned}\rho_g(-X) &= \int_0^{-\infty} [g(S_{-X}(x) - 1] \, dx + \int_0^\infty g(S_{-X}(x)) \, dx \\ &= \int_{-\infty}^u [g(1 - S_X(-x) + P[X = x]) - 1] \, dx + \int_{-\infty}^u g(1 - S_X(-x) + P[X = x]) \, dx \\ &= \int_{-\infty}^u (g(1 - S_X(-x)) - 1) \, dx + \int_{-\infty}^u g(1 - S_X(-x)) \, dx.\end{aligned}$$

Replacing x by $-u$, we get

$$\rho_g(-X) = -\int_{-\infty}^u (g(1 - S_X(u)) - 1) \, du - \int_{-\infty}^u g(1 - S_X(u)) \, du = -\rho_{\tilde{g}}(X).$$

7. In the generalized case, distortion risk measures are not additive[18]:

$$\rho_g(X+Y) \neq \rho_g(X) + \rho_g(Y)$$

8. Distortion risk measures are sub-additive if and only if the distortion function $g(x)$ is concave.

$$\rho_g(X+Y) \leq \rho_g(X) + \rho_g(Y)$$

The proof is given in Wirch and Hardy (1999). Hence, concave distortion risk measures are coherent risk measures.

9. For a non-decreasing distortion function g, the associated risk measure ρ_g is consistent with the stochastic dominance of order 1

$$X \leq_1 Y \Rightarrow \rho_g(X) \leq \rho_g(Y)$$

The proof is given in Hardy and Wirch (2003).

10. For a non-decreasing concave distortion function g, the associated risk measure ρ_g is consistent with the stochastic dominance of order 2 (i.e., SSD)

$$X \leq_2 Y \Rightarrow \rho_g(X) \leq \rho_g(Y).$$

As a result, every coherent distortion risk measure is consistent with respect to second-order stochastic dominance.

11. For a strictly concave distortion function g, the associated risk measure ρ_g is strictly consistent with the stochastic dominance of order 2

$$X <_2 Y \Rightarrow \rho_g(X) < \rho_g(Y)$$

The proof is given in Hardy and Wirch (2003).

12. Consistency of distortion risk measures with respect to the higher-order stochastic dominances was analyzed in the financial and actuarial literature. In particular, Hürlimann (2004) obtained some results about the consistency of distortion risk measures with stochastic dominance of order 3. The necessary precondition to that is the consistency with respect to 3-convex order. The only distortion risk measures which are consistent with 3-convex order are $g(x) = \sqrt{x}$ and $g(x) = x$ under the assumption that the set of possible losses contains all Pareto[19] variables (Theorem 6.3 in Hürlimann (2004)). Under a much weaker hypothesis of discrete losses, Bellini and Caperdoni (2006) showed that the only coherent distortion risk measure that is consistent with respect to the 3-convex order is the expected value, when $g(x) = x$, leaving the problem open for the case of continuous losses.

[18] The proof is as follows. Consider the function $g = x^2$, the joint distribution of discrete random variables X and Y is defined as follows: $P(1,1) = P(-1,1) = P(1,-1) = P(-1,-1) = 0.25$. The marginal distributions X and Y have the forms: $P(1) = P(-1) = 0.5$. Direct calculations show that

$$\rho_g(X) = \rho_g(Y) = -0.5; \quad \rho_g(X+Y) = 0.25; \quad 0.25 \neq -1$$

The risks X and Y are independent here.

[19] With a generic location and scale parameter allowed.

5 Examples of distortion risk measures

As explained above, the choice of distortion function specifies the distortion risk measures. Thus, finding "good" distorted risk measures boils down to the choice of a "good" distortion function. The properties one might use as a criteria for the choice of a distortion function include *continuity*, *concavity*, and *differentiability*. Many different distortions g have been proposed in the literature. Some well-known ones are presented below. A summary of other proposed distortion functions can be found in Denuit et al. (2005).

❏ With $g(x) = x$, we have $\rho_g(X) = E[X]$, if the mathematical expectation exists[20].

❏ VaR corresponds to the distortion:

$$g(x) = \begin{cases} 0, & \text{if } x < 1-p; \\ 1, & \text{if } x \geq 1-p. \end{cases}$$

The distortion function is discontinuous in this case due to the jump at $x = 1 - p$ (see Figure 1). This predetermines that VaR is not coherent. As a result, VaR does not represent a "good" behaved distortion function.

❏ CVaR can be defined as a distortion risk measure based on the distortion function

$$g(x) = \min\left(\frac{x}{1-p}, 1\right), \quad x \in [0,1]$$

Figure 2 presents the given function. It is continuous, implying that CVaR is coherent. But the distortion function of CVaR is not differentiable at $x = 1 - p$. Consequently, it discards potentially valuable information because it maps all percentiles below $(1-p)$ to a single point "0". By doing so, it fails to take into account the severity of extreme values (Wang (2002)).

❏ In order to overcome these sorts of problems, Wang (2002) considers the following specification of g:

$$g(x) = \Phi\left(\Phi^{-1}(x) - \Phi^{-1}(q)\right),$$

[20] We limit our proof to the interval $[-a, a]$. In this case the mathematical expectation accurately exists.

$$\begin{aligned}\rho_g(X) &= \int_0^{-a} (S_X(x) - 1)\,dx + \int_a^0 S_X(x)\,dx \\ &= -\int_{-a}^0 F_X(x)\,dx + \int_0^a (1 - F_X(x))\,dx \\ &= a - \int_{-a}^a F_X(x)\,dx\end{aligned}$$

By integrating by parts, we get

$$\rho_g(X) = a - xF_X(x)|_{-a}^a + \int_{-a}^a x\,dF_X(x) = E[X],$$

as $F_X(a) = 1$, $F_X(-a) = 0$. In this particular case, the distortion risk measures are additive.

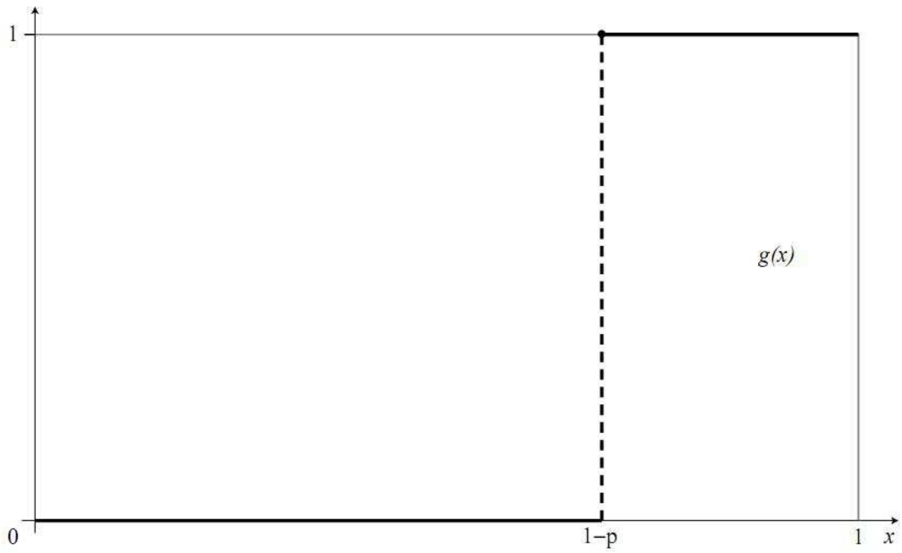

Figure 1. Distortion function of VaR

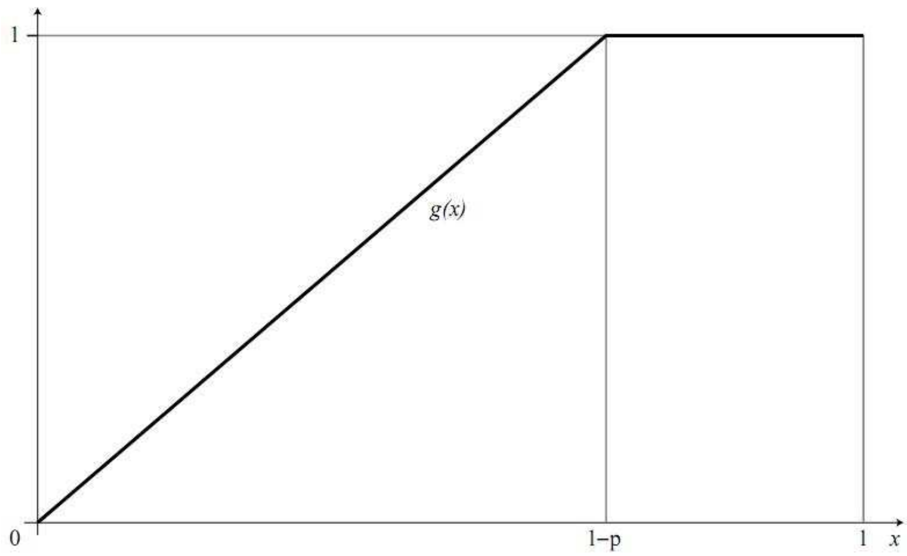

Figure 2. Distortion function of CVaR

for $p \in [0,1]$, where $0 < q \leq 0.5$ is some parameter[21]. The distortion function g is indeed non-decreasing, concave, and such that $g(0) = 0$ and $g(1) = 1$. The corresponding risk measure WT_q is known as the *Wang transform*. The parameter q can be changed to make the Wang transform either sharper on high losses or softer and more receptive to positive returns. Wang (2002) recommended the Wang transform for the measurement of insurance risks.

❏ The beta family of distortion risk measures, proposed by Wirch and Hardy (1999), utilizes the incomplete beta function:

$$g(F_X(x)) = \beta(a,b;F_X(x)) = \int_0^{F_X(x)} \frac{1}{\beta(a,b)} t^{a-1}(1-t)^{b-1}\,dt = S_\beta(F_X(x))$$

where $S_\beta(x)$ is the distribution function of the beta distribution, and $\beta(a,b)$ is the beta function with parameters $a > 0$ and $b > 0$, that is

$$\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt.$$

The beta-distortion risk measures are concave if and only if $a \leq 1$ and $b \geq 1$; strictly concave if a and b are both not equal to 1.

❏ The *Proportional Hazard* (PH) *transform* is a special case of the beta-distortion risk measure with $a = 0.1$, $b = 1$. The PH-transform risk measure is defined as:

$$\rho_{PH}(X) = \int_0^\infty S_X(x)^{\frac{1}{\gamma}}\,dx, \quad \gamma > 1,$$

where $S_X(x) = 1 - F_X(x)$.

6 New distortion risk measures

The notion of asymmetry of the risk perception of investors was studied in the classical works such as Kahneman and Tversky (1979). Here we apply the idea of asymmetry to the standard construction of the distortion risk measures. Introducing the asymmetry into the Choquet integral, we obtain the following distortion risk measures:

$$\rho_{g_i}(X) = -\int_{-\infty}^0 [1 - g_1(S_X(x))]\,dx + \int_0^\infty g_2(S_X(x))\,dx,$$

where distortion functions g_1 and g_2 only differ by their risk-averse parameters.

Moreover, the risk-averse parameter should be inserted in the properly chosen distortion function. Here we propose the quadratic distortion function

$$g_i(x) = x + k_i(x - x^2),$$

[21] We need $q \leq 0.5$ in order to get concave distortion risk measures.

where $k_i \in (0,1]$ is the risk-averse or concave parameter. The investor is more risk-averse with k closer to one. The chosen quadratic distortion function $g\colon [0,1] \to [0,1]$ comes within all the criteria of a "good" distortion function: it is continuous, differentiable, and strictly concave[22] when $k \in (0,1]$. A strictly concave function leads to consistency with respect to SSD. In the case when $k = 0$, the distortion function equals the mathematical expectation $g(x) = x$.

As a result, we obtain a new asymmetric distortion risk measure based on the quadratic distortion function:

$$\rho_{g_i}(X) = -\int_{-\infty}^{0} [1 - g_1(S_X(x))]\, dx + \int_{0}^{\infty} g_2(S_X(x))\, dx.$$

where $g_i(x) = x + k_i(x - x^2)$, $k_i \in (0,1]$, and $i = 1,2$, k_1, k_2 are changing independently. The proposed risk measure treats upside and downside risk differently. The motivation for the introduction of asymmetry is the importance for the risk-averse investor of having $k_1 > k_2$ in order to put more weight on the left tail (losses), than on the right (gains).

The power function is widely used in economic theory, that is why it seems to be promising to use this function in the proposed framework of asymmetric distortion risk measures. We will apply the following form of the power distortion function:

$$g = x^k, k \in (0,1).$$

Risk-averse investors using the power distortion function to describe their risk perception will choose k closer to 0. The asymmetric distortion risk measured with the power distortion function will take the form:

$$\rho_{g_i}(X) = -\int_{-\infty}^{0} [1 - g_1(S_X(x))]\, dx + \int_{0}^{\infty} g_2(S_X(x))\, dx.$$

where $g_i(x) = x_i^k$, $k_i \in (0,1)$, $i = 1,2$, k_1, k_2 are changing independently. We again include the asymmetry by introducing different parameters on the left and right sides of the integral.

7 Summary

The natural question that arises for asset managers is the choice of an adequate risk measure. The answer to this question is not obvious, as it is generally not easy to identify which particular risk measure might be the best and there is no clear way of comparing one risk measure to another. Furthermore, there is no guarantee that an arbitrarily chosen measure would necessarily be "good".

In the paper, the class of distortion risk measures is analyzed. It possesses the most desirable properties for a portfolio risk measure: law-invariance, sub-additivity, and consistency with the second order stochastic dominance. In addition, distortion risk measures

[22] $g''(x) = -2k < 0$ at all points.

have their roots in the distortion utility theory of choice under uncertainty, meaning that this class of risk measures can better reflect the risk preferences of investors.

The well-known examples of distortion risk measures were reviewed and the drawbacks of VaR and CVaR in the capacity of distortion function were explained. Moreover, we introduce new asymmetric distortion risk measures that possess the property of asymmetry along with the standard properties of concave distortion risk measures. This new risk measures reflects the whole range of an investor's preferences.

8 Appendix - Properties of risk measures

Axioms to characterize a particular risk measure are usually necessary to obtain mathematical proofs. They can generally be divided into three types (Denuit et al. (2006)):

- *Basic rationality axioms* are satisfied by most of the risk measures (e.g., monotonicity);
- *Additivity axioms* include sums of risks (e.g., sub-additivity, additivity, and super-additivity);
- *Technical axioms* deal mostly with continuity conditions.

None of the following properties is absolute. Almost all of them are subject to criticism.

Property 1. Law-invariance

Law-invariance states that a risk measure $\rho(X)$ does not depend on a risk itself but only on its underlying distribution, i.e. $\rho(X) = \rho(F_X)$, where F_X is the distribution function of X. This condition ensures that F_X contains all the information needed to measure the riskiness of X. Law-invariance can be phrased as:

$$F_X = F_Y, \Rightarrow \rho(X) = \rho(Y)$$

for every random portfolio returns X and Y with distribution functions F_X and F_Y. In other words, ρ is law-invariant in the sense that $\rho(X) = \rho(Y)$, whenever X and Y have the same distribution with respect to the initial probability measure, P. This assumption is essential for a risk measure to be estimated from empirical data, which ensures its applicability in practice.

Property 2. Positive homogeneity

Positive homogeneity (also known as positive scalability) formulates as follows: for each positive λ and random portfolio return $X \in \mathcal{X}$:

$$\rho(\lambda X) = \lambda^k \rho(X).$$

Positive homogeneity signifies that a measure has the same dimension (scalability) as a variable X. When the parameter $k = 0$, a risk measure does not depend on the scalability.

From a financial perspective, positive homogeneity implies that a linear increase of the return by a positive factor leads to a linear increase in risk by the same factor. Although Artzner et al. (1999) require adequate risk measures to satisfy the positive homogeneity property, Föllmer and Schied (2002a, 2002b) drop this assumption arguing that risk may grow in a non-linear way as the size of the position increases. This would be the case with liquidity risk. Dhaene et al. (2003) and de Giorgi (2005) also do not believe that this rule characterizes rational decision-makers' perception of risk.

Property 3. Sums of risks

Consider two different financial instruments with random payoffs $X, Y \in \mathcal{X}$. The payoff of a portfolio consisting of these two instruments will equal $X + Y$.

Property 3.1. Sub-additivity

Sub-additivy states that the risk of the portfolio is not greater than the sum of the risks of the portfolio components. In other words, "a merger does not create extra risk" (Artzner et al. (1999)).

$$\rho(X + Y) \leq \rho(X) + \rho(Y)$$

Compliance with this property tends to the diversification effect. Though Artzner et al. (1999) treat sub-additivity as a necessary requirement for constructing a risk measure in order for it to be coherent, empirical evidence suggests that sub-additivity does not always hold in reality[23].

Property 3.2. Additivity

The additivity property is expressed in the following form:

$$\rho(X + Y) = \rho(X) + \rho(Y)$$

This property is valid for independent and comonotonic[24] random variables X and Y. The comonotonic random variables with no-hedge condition result in *comonotonic additivity*.

Property 3.3. Super-additivity

Super-additivity states that the portfolio risk estimate could be greater than the sum of the individual risk estimates.

$$\rho(X + Y) \geq \rho(X) + \rho(Y)$$

The super-additivity property is valid for risks which are positive (negative) dependent.

Property 4. Convexity

(1) For all $X, Y \in \mathcal{X}$, $0 \leq \lambda \leq 1$, the following inequality is true:

$$\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y).$$

Convexity ensures the diversification property and relaxes the requirement that a risk measure must be more sensitive to aggregation of large risks.

[23] Critiques of sub-additivity can be found in Dhaene et al. (2003) and Heyde et al. (2006).

[24] Comonotonic or common monotonic random variables (Yaari (1987), Schmeidler (1986), Dhaene et al.(2002a, 2002b)) are those such that if the increase of one follows the increase of the other variable:

$$P[X \leq x, Y \leq y] = \min\{P[X \leq x], P[Y \leq y]\} \text{ for all } x, y \in \mathbb{R}.$$

Intuitively, such variables have a maximal level of dependency. Comonotonic random variables are necessarily positively correlated. In financial and insurance markets, this property appears quite frequently.

(2) For any $\lambda, \mu \geq 0$, $\lambda + \mu = 1$, and distribution functions F, G, the following inequality holds

$$\rho(\lambda F + \mu G) \leq \lambda \rho(F) + \mu \rho(G).$$

(3) *Generalized convexity.* For any $\lambda, \mu \geq 0$, $\lambda + \mu = 1$ and distribution functions U, V, H, such that the following random variables exist X, Y, $\lambda X + \mu Y$, for which $F_X = U$, $F_Y = V$, $F_{\lambda X + \mu Y} = H$, the inequality is true

$$\rho(H) \leq \lambda \rho(U) + \mu \rho(V).$$

Property 5. Monotonicity

For every random portfolio returns X and Y such that $X \geq Y$,

$$\rho(X) \leq \rho(Y).$$

Monotonicity implies that if one financial instrument with the payoff X is not less than the payoff Y of the other instrument, then the risk of the first instrument is not greater than the risk of the second financial instrument. Another presentation of the monotinicy property with a risk-free instrument is as follows:

$$X \geq 0 \Rightarrow \rho(X) \leq \rho(0)$$

for $X \in \mathcal{X}$.

Property 6. Translation invariance

Property 6.1. For the non-negative number $\alpha \geq 0$ and $C \in \mathbb{R}$, the property has the following form:

$$\rho(X + C) = \rho(X) - \alpha C.$$

This property states that if the payoff increases by a known constant, the risk correspondenly decreases. In practice, $\alpha = 0$ or $\alpha = 1$ are often used.

Property 6.2. When $\alpha = 0$, it implies that the addition of a certain wealth does not increase risk. This property is also known as the *Gaivoronsky-Pflug (G-P) translation invariance* (Gaivoronski and Pflug (2001)).

Property 6.3. The case when $\alpha = 1$ implies that by adding a certain payoff, the risk decreases by the same amount.

$$\rho(X + C) = \rho(X) - C.$$

Property 6.4. When a constant wealth has a positive value, i.e., $C \geq 0$, one gets

$$\rho(X + C) \leq \rho(X).$$

This result is in agreement with the monotonicity property of $X + C \geq X$.

Property 6.5. In particular, translation invariance involves

$$\rho\left(X+\rho(X)\right)=\rho(X)-\rho(X)=0,$$

obtaining a risk-neutral position by adding $\rho(X)$ to the initial position X.

Property 7. Consistency

Property 7.1. Consistency with respect to *n-order stochastic dominance* has the following general form:

$$X \geq_n Y, \rho(X) \geq \rho(Y).$$

In practice, the maximal value of $n = 2$; $n = 0$ just stands for a monotonicity property.

Property 7.2. *Monotonic dominance of n-order*

$$X \geq_{M(n)} Y, \text{ iff } E[u(X)] \geq E[u(Y)]$$

for any monotonic of order n functions, that is $u^{(n)}(t) \geq 0$.

It is known, that $X \geq_1 Y$ is equivalent to $X \leq_{M(1)} Y$. $X \leq_{M(2)} Y$ is also called the Bishop-de Leeuw ordering or Lorenz dominance.

Property 7.3. *First-order stochastic dominance* (FSD)

$$\text{For } X \geq_1 Y,\ F_X(x) \leq F_Y(x)$$

If an investor prefers X to Y, then FSD will indicate that the risk of X is less than the risk of Y. In terms of utility function u, the following holds

$$\text{If } X \geq_1 Y,\ \text{then } E[u(X)] \geq E[u(Y)]$$

for all increasing utility functions u. FSD characterizes the preferences of risk-loving investors. Ortobelli et al. (2006) classified risk measures consistent with respect to FSD as a *safety-risk measures*[25].

Property 7.4. *Rothschild-Stiglitz stochastic order dominance* (RSD)

RSD was introduced by Rothschild and Stiglitz (1970) and has the form:

$$\text{If } X \leq_{RS} Y,\ \text{then } E[u(X)] \geq E[u(Y)]$$

for any concave, not necessarily decreasing, utility function u. RSD describes preferences of risk-averse investors. *Dispersion measures* are normally consistent with RSD.

[25] In the portfolio selection literature, two disjoint categories of risk measures are defined: dispersion measures and safety-first risk measures. For the definitions and properties of specified categories, see for example, Giacometti and Ortobelli (2004).

Property 7.5. *Second-order stochastic dominance* (SSD)

The concept of SSD was introduced by Hadar and Russell (1969), although Rothschild and Stiglitz (1970) first proposed its use in portfolio theory. SSD has the following form:

$$\text{For } X \geq_2 Y,\ E[u(X)] \geq E[u(Y)]$$

for all increasing, concave utility functions u. SSD characterizes non-satiable risk-averse investors.

Property 7.6. *Stochastic order - stop-loss*

Y dominates X ($Y \geq_{SL} X$) in stop-loss order, if for any number α the following inequality is true:

$$E[(Y-\alpha)^+] \geq E[(X-\alpha)^+].$$

Here $\alpha^+ = \max\{0, \alpha\}$. Such order is essential in the insurance industry. If the insurer takes the responsibility for the claims greater than α (deductible), then the expected claim Y is not smaller than X.

Property 7.7. *Convex order*

Y dominates X with respect to convex order ($Y \geq_{CX} X$), if the relation $Y \geq_{SL} X$ is true and when $\alpha = -\infty$ in stop-loss order, i.e. $E[X] = E[Y]$. Convex ordering is related to the notion of risk aversion[26].

Consistency with the stochastic dominance is a necessary property for a risk measure, because it enables one to characterize the set of all optimal portfolio choices when either wealth distributions or expected utility functions depend on a finite number of parameters (Ortobelli (2001)).

Property 8. Non-negativity

Property 8.1. $\rho(X) \geq 0$, while $\rho(X) > 0$ for all non-constant risk.

Property 8.2. If $X \geq 0$, then $\rho(X) \leq 0$; if $X \leq 0$, then $\rho(X) \geq 0$.

Property 9. Continuity

Property 9.1. Probability convergence continuity: If $X_n \xrightarrow{\text{P}} X$, then $\rho(X_n)$ converges and has the limit $\rho(X)$.

Property 9.2. Weak topology continuity: If $F_X \xrightarrow{\text{w}} F_X$, then $\rho(F_{X_n})$ converges and has a limit $\rho(F_X)$.

Property 9.3. Horizontal shift continuity: $\lim_{\delta\to 0} \rho(X+\delta) = \rho(X)$.

Property 9.4. Opportunity of arbitrary risk approximation with the finite carrier

[26] See also Kaas et al. (1994, 2001).

is expressed by the equality[27]:

$$\lim_{\delta\to+\infty}\rho(\min\{X,\delta\}) = \lim_{\delta\to-\infty}\rho(\max\{X,\delta\}) = \rho(X).$$

Property 9.5. Lower semi-continuity: For any $C \in \mathbb{R}$, the set $\{X \in \mathcal{X} : \rho(X) \le C\}$ is $\sigma(L^\infty, L^1)$ - closed.

Property 9.6. Fatough property[28]
For any bounded sequence (X_n) for which $X_n \xrightarrow{\text{P}} X$, the following holds:

$$\rho(X) \le \lim_{n\to\infty}\inf \rho(X_n).$$

These properties are cardinally important. Nonfulfilment of the continuity property implies that even a small inaccuracy in a forecast can lead to the poor performance of a risk measure.

Property 10. Strictly expectation-boundedness

The risk of a portfolio is always greater than the negative of the expected portfolio return.

$$\rho(X) \ge -E[X],\ \text{while}\ \rho(X) > -E[X]\ \text{for all non-constant}\ X,$$

where $E[X]$ is the mathematical expectation of X.

Property 11. Lower-range dominated

Deviation measures possess lower-range dominated property of the following form:

$$D(X) \le E(X)$$

for a non-negative random variable. From *property 10* and *property 11* one can derive:

$$D(X) = \rho(X - EX), \quad \rho(X) = D(X) - E(X)$$

Property 12. Risk with risk-free return C

Property 12.1. $\rho(C) = -C$, it follows from the *invariance property 6.3.* If $C > 0$, then the situation is stable, risk is negative. The opposite situation occurs with $C < 0$.

Property 12.2. $\rho(C) = 0$, risk does not deviate with the zero certain return.

According to the classification given by Albrecht (2004), a number of risk measures can be divided into two categories - measures of the “first kind” and “second kind” - subject to the type of risk conception. Risk measures with *property 2* and *property 12.2* belong to the “first kind”, where risk is perceived as the quantity of deviations from a target. Risk measures with *property 2* and *property 12.1* are of the “second kind”, where risk is considered as a “necessary capital respectively necessary premium”.

[27]Wang et al. (1997), Hürlimann (1994).
[28]In some contexts, it is equivalent to the upper semi-continuity condition with respect to $\sigma(L^\infty, L^1)$.

Property 13. Symmetric property

(1) $\rho(-X) = -\rho(X)$, which corresponds to *property 8.1*.

(2) $\rho(-X) = \rho(X)$, this property makes sense for the measures with posssible negative values (*property 8.2* fulfilled).

Property 14. Allocation

A risk measure need not be defined on the whole set of values of a random variable. Formally, in a given set U, from the condition $F_X = F_Y$, when $x \notin U$, it follows that $\rho(X) = \rho(Y)$. Apparently, this property holds only for law-invariant measures. Most often, some threshold value T is assigned, and the set U takes values $U = (-\infty, T]$ or $U = [T, \infty)$.

Property 15. Static and dynamic natures

It is useful to use a dynamic and multi-period framework to answer the following question: How should an institution proceed with new information in each period and how should it reconsider the risk of the new position? Riedel (2004) introduced the specific axioms such as predictable translation invariance and dynamic consistency for a risk measure to capture the dynamic nature of financial markets.

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