Barrier Option Pricing by Branching Processes

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Abstract

This paper examines the pricing of barrier options when the price of the underlying asset is modeled by branching process in random environment (BPRE). We derive an analytical formula for the price of an up-and-out call option, one form of a barrier option. Calibration of the model parameters is performed using market prices of standard call options. Our results show that the prices of barrier options that are priced with the BPRE model deviate significantly from those modeled assuming a lognormal process, despite the fact that for standard options, the corresponding differences between the two models are relatively small.

Key words: Barrier option, up-and-out call option, Bienayme-Galton-Watson branching process, branching process in random environment

1 Introduction

Barrier options, also known as knock-in options or knock-out options, have become increasingly popular since they were first traded in the over-the-counter (OTC) market in 1967. Their popularity resulted in the introduction of barrier options on exchanges by 1991, first by the Chicago Board Option Exchange and then by the American Stock Exchange. The appealing feature of this exotic option product is twofold. First, it provides an investor with flexibility in a number of hedging strategies. Second, it is less expensive than standard options.

Barrier options are a type of path-dependent option which include down-and-out options, down-and-in options, up-and-out options, and up-and-in options. In 1973, Merton [20] provided a closed-form solution for down-and-out call options. Since then, closed-form solutions for many European barrier options have been developed by Reiner and Rubinstein [23], Kunitomo and Ikeda [15], Carr [8], and Geman and Yor [14]. Numerical methods have been proposed to handle American or other more complex barrier options by Boyle and Lau [4], Ritchken [25] , Broadie, Glasserman, and Kou [7], Boyle and Tian [6], and Zvan, Vetzal and Forsyth [31]. Most of the published papers on barrier options assume the underlying asset follows geometric Brownian motion. This implies that the price of the underlying asset is log-normally distributed as in the Black-Scholes [3] model. This assumption has several drawbacks.

The first drawback is that stock prices on the New York Stock Exchange (NYSE) were quoted in units of \$1/8, then in \$1/16, and now \$0.01. This discreteness of the stock price contradicts the continuous distribution assumption in the Black-Scholes formula. Second, it is evident that stock prices sometimes exhibit large jumps when some important news is disclosed. Third, extensive empirical evidence, pioneered by Mandelbrot [18, 19] and Fama [13], empirically documented that the logarithm of stock returns tend to be leptokurtic; that is, their distributions have thicker tails than the normal distribution derived from the geometric Brownian motion law. Furthermore, Black [2] noted the so-called "leverage effect," meaning that the volatility of stock returns tends to be negatively correlated with the price.

There are different ways to avoid these drawbacks of the lognormal model. One of them is to model the price of the underlying asset using different stochastic processes which possess some of the stylized facts reported for stock prices. Thus, Zhou [30] examines the case where the stock price follows a jump diffusion process. Valuation of barrier options in a constant elasticity of variance (CEV) model is treated in Boyle and Tian [5] and Davydov and Linetsky [10]. Shoutens and Symens [27] utilize Levy stochastic volatility models to price barrier options. Riberio and Webber [24] present a Monte Carlo algorithm for pricing barrier options with the variance gamma (VG) model.

In the present paper, we use a branching process in a random environment (BPRE) to model the stock price. This model of stock price movement was introduced by Epps [11] in 1996. The model, constructed by Bienayme-Galton-Watson branching process subordinated with a Poisson process, captures the stylized facts about stock return distributions. Specifically, the return distribution exhibits thick tails and a variance that decreases with the level of the stock price. The process also allows for possible jumps in stock prices, and takes into account the possibility of bankruptcy, something that is neglected in many other models.

Williams [29] provides an exact formula for the price of a European put option based on the BPRE model. Liu [17] applies the BPRE model to options on a sample of individual U.S. equities. Inferring the parameters from transaction prices of traded options, Liu finds that for the in-sample prediction, the model typically eliminates the smile effect that has been observed for option prices. The results are mixed when out-of-sample predictions are compared with the ad hoc version of Black-Scholes with moneyness-specific implicit volatilities. Liu does find evidence that the predictions are better for options on low-priced stocks, where the discreteness arising from the minimum tick size would seem to be more relevant.

In this paper, we derive a formula for the price of an up-and-out call option based on BPRE and compare the results with the prices based on the lognormal model. The paper is organized as follows. In Section 2, we review the mathematical definition of the BPRE model and its main properties and advantages. Since we rely on a methodology for the pricing of a European call option formulated by Mitov and Mitov [21] for the calibration of the equivalent martingale measure (EMM) parameters, we provide the formula at the end of Section 2. The formula for the price of an up-and-out call option is derived in Section 3. In Section 4, we report numerical results for barrier options on several stocks and the S&P 500 index. We calibrate both BPRE and lognormal models to market prices of standard options, and subsequently price up-and-out call options. We then compare those prices based on the BPRE model and lognormal model. Section 5 summarizes our findings.

2 The Model

2.1 Branching Processes

Let us consider a Bienayme-Galton-Watson branching process, Z_n, $n = 0, 1, 2, \ldots$ with a non-random number of ancestors $Z_0 > 0$, and the offspring probability distribution

$$\begin{aligned}
&\mathbb{P}(Z_{n+1} = 0 | Z_n = 1) = (1 - a), \\
&\mathbb{P}(Z_{n+1} = k | Z_n = 1) = ap(1 - p)^{k-1}, k = 1, 2, \ldots, \\
&0 < a < 1, \quad 0 < p < 1.
\end{aligned}$$

The probability generating function (p.g.f.) $f(s) = \mathbb{E}[s^{Z_1} | Z_0 = 1]$ of the offspring distribution can be easily represented in terms of the distribution's factorial moments by

$$f(s) = 1 - \frac{m(1-s)}{1 + \dfrac{b}{2m}(1-s)}, \quad s \in [0, 1],$$

where $m = a/p$ is the offspring mean and

$$b = 2\frac{1-p}{p}m, \quad \sigma^2 = b + m - m^2 = \frac{a}{p}\left(\frac{(1-p) + (1-a)}{p}\right)$$

are the offspring second moment and the offspring variance.

Such a function has fractional linear form. It is well known (see e.g. Sevastyanov [26]) that for the p.g.f. $f_n(s) = \mathbb{E}[s^{Z_n} | Z_0 = 1]$, the following relation holds

$$f_n(s) = 1 - \frac{m^n(1-s)}{1 + \dfrac{b}{2m}\dfrac{1-m^n}{1-m}(1-s)}. \tag{1}$$

In Section 3, we will need the probabilities

$$\mathbb{P}(Z_n = k | Z_0 = 1), \quad k = 0, 1, 2, \ldots, \quad n = 1, 2, \ldots.$$

Differentiating (1), we obtain

$$f_n'(s) = \frac{m^n}{\left[1 + \frac{b(1-m^n)}{2m(1-m)}(1-s)\right]^2}, \tag{2}$$

and for $k = 2, 3, \ldots,$

$$f_n^{(k)}(s) = \frac{k!m^n \left[\frac{b(1-m^n)}{2m(1-m)}\right]^{k-1}}{\left[1 + \frac{b(1-m^n)}{2m(1-m)}(1-s)\right]^{k+1}}. \tag{3}$$

From the properties of the p.g.f., it follows that

$$\mathbb{P}(Z_n = k | Z_0 = 1) = \frac{f_n^{(k)}(0)}{k!},$$

therefore we get for $k = 1, 2, \ldots$

$$\mathbb{P}(Z_n = k | Z_0 = 1) = \frac{m^n \left[\frac{b(1-m^n)}{2m(1-m)}\right]^{k-1}}{\left[1 + \frac{b(1-m^n)}{2m(1-m)}\right]^{k+1}}. \tag{4}$$

Then the probability for extinction in the $n-$th generation is

$$\begin{aligned} \mathbb{P}(Z_n = 0 | Z_0 = 1) &= 1 - \sum_{k=1}^{\infty} \frac{m^n \left[\frac{b(1-m^n)}{2m(1-m)}\right]^{k-1}}{\left[1 + \frac{b(1-m^n)}{2m(1-m)}\right]^{k+1}} \\ &= 1 - \frac{m^n}{1 + \frac{b(1-m^n)}{2m(1-m)}}. \end{aligned} \tag{5}$$

Substituting $s = 1$ in (2), we get the first moment of the process

$$\mathbb{E}[Z_n | Z_0 = 1] = \begin{cases} m^n, & m \neq 1, \\ 1, & m = 1. \end{cases} \tag{6}$$

Substituting $k = 2$ and $s = 1$ in (3), we obtain the second factorial moment

$$\mathbb{E}[Z_n(Z_n - 1) | Z_0 = 1] = \begin{cases} bm^{n-1}\frac{1-m^n}{1-m}, & m \neq 1, \\ bn, & m = 1. \end{cases} \tag{7}$$

If the process starts with $Z_0 = I > 1$ number of ancestors, it is a sum of I independent and identically distributed branching processes, each of them beginning with an ancestor. Therefore,

$$\mathbb{E}[s^{Z_n} | Z_0 = I] = (f_n(s))^I$$

and we can calculate for $k = 1, 2, \ldots$

$$\mathbb{P}(Z_n = k | Z_0 = I) = \frac{d^k (f_n(s))^I}{ds^k}\bigg|_{s=0}.$$

Instead of using the derivatives, we suggest the following simple iterative procedure. For any $n > 0$, $Z_0 = I > 1$, and $K \geq 0$, calculate:

(i) $\mathbb{P}(Z_n = k | Z_0 = 1)$ for $k = 0, 1, 2, \ldots, K$ by (4) and (5).

(ii) For $Z_0 = 2$, the process is the sum of two independent processes each of which begins with $Z_0 = 1$ particle. Therefore,

$$\mathbb{P}(Z_n = k | Z_0 = 2) = \sum_{j=0}^{k} \mathbb{P}(Z_n = j | Z_0 = 1)\mathbb{P}(Z_n = k - j | Z_0 = 1),$$

for all $k = 0, 1, 2, \ldots, K$.

(iii) For $Z_0 = I, \quad I \geq 3$, the process is the sum of two independent processes one of which starts with $Z_0 = I - 1$ and the other starts with $Z_0 = 1$ particle. That is,

$$\mathbb{P}(Z_n = k | Z_0 = I) = \sum_{j=0}^{k} \mathbb{P}(Z_n = j | Z_0 = I - 1)\mathbb{P}(Z_n = k - j | Z_0 = 1).$$

for all $k = 0, 1, 2, \ldots, K$.

Note that the calculations can be easily performed using numerical software if for a fixed n and K, the probabilities $p_{1k}(n) = \mathbb{P}(Z_n = k | Z_0 = 1)$, $k = 0, 1, 2, \ldots, K$, calculated in the point (i) are entered in an upper triangular matrix as follows

$$\mathbf{P}(n) = \begin{pmatrix} p_{10}(n) & p_{11}(n) & p_{12}(n) & \ldots & p_{1K}(n) \\ 0 & p_{10}(n) & p_{11}(n) & \ldots & p_{1,K-1}(n) \\ 0 & 0 & p_{10}(n) & \ldots & p_{1,K-2}(n) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & p_{10}(n) \end{pmatrix}.$$

It is not difficult to see that the probabilities $p_{Ik}(n) = \mathbb{P}(Z_n = k | Z_0 = I)$, $k = 0, 1, 2 \ldots, K$ are equal to the $(1, k)-$th element of the $I-$th power of matrix $\mathbf{P}(n)$.

2.2 Branching process in random environment as a price process

Assume now that on the common probability space $(\Omega, \mathcal{A}, \mathbb{P})$ are given (i) a supercritical $(m > 1)$ Bienayme-Galton-Watson branching process $Z_n, n = 0, 1, 2, \ldots$ defined in the previous section and (ii) an independent of it Poisson process $N(t), \ t \geq 0$ with intensity $\lambda > 0$. Following Epps [11] or [12], define the randomly indexed branching process, which is in fact a branching process in random environment (BPRE),

$$S(t) = Z_{N(t)},\ t \geq 0.$$

Here $S(t)$ represents the price of one share of stock at time t measured in units of minimum price movements (for example \$0.01). Equity prices are then viewed as consisting of an integer number of "price particles." In each period, each "price particle" of equity price produces a random number of offspring "price particles," the aggregate number of which comprises the equity price in the next period. Hence, by allowing a random number of generations to occur in each period, the BPRE model generates prices in continuous time.

The independence of Z_n and $N(t)$ yields the following formula for the p.g.f. of the process $S(t) = Z_{N(t)}$, starting with $Z_0 \equiv S(0) \geq 1$ ancestors

$$\begin{aligned}\Phi(t,s) &= \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \left(f_n(s)\right)^{S(0)} \\ &= \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \left(1 - \frac{m^n(1-s)}{1 + \frac{b}{2m}\frac{1-m^n}{1-m}(1-s)}\right)^{S(0)}.\end{aligned}$$

Using the p.g.f. and formulas (6) and (7), after some calculations we derive the following formulas for the mean and the variance of the process $S(t)$

$$M(t) = \mathbb{E}[S(t)|S(0)] = \begin{cases} S(0)e^{\lambda t(m-1)}, & m \neq 1, \\ S(0), & m = 1, \end{cases} \tag{8}$$

$$\sigma^2(t) = Var[S(t)|S(0)] = \begin{cases} S(0)^2[e^{\lambda t(m^2-1)} - e^{2\lambda t(m-1)}] \\ +\dfrac{S(0)\sigma^2[e^{\lambda t(m^2-1)} - e^{\lambda t(m-1)}]}{m(m-1)}, & m \neq 1 \\ S(0)\sigma^2\lambda t, & m = 1. \end{cases}$$

These formulas allow us to examine in detail some of the main properties of the total return $R(t) = [S(t) - S(0)]/S(0)$ over a period $(0, t)$. The first two moments of the return distribution have the following form:

$$\mathbb{E}[R(t)|S(0)] = \begin{cases} e^{\lambda t(m-1)} - 1, & m \neq 1 \\ 0, & m = 1. \end{cases}$$

$$Var[R(t)|S(0)] = \begin{cases} e^{\lambda t(m^2-1)} - e^{2\lambda t(m-1)} \\ +\dfrac{1}{S(0)}\left(\dfrac{\sigma^2}{m(m-1)}(e^{\lambda t(m^2-1)} - e^{\lambda t(m-1)})\right), & m \neq 1 \\ \dfrac{b\lambda t}{S(0)}, & m = 1. \end{cases}$$

The coefficient of $1/S(0)$ in the variance representation is positive, because we examine a supercritical Bienayme-Galton-Watson branching process, that is $m > 1$. Therefore, the variance of the return is inversely related to the stock price. As a result, the so-called "leverage effect" is built into the model.

Formulas for the skewness $\gamma_1[R(t)]$ and kurtosis $\gamma_2[R(t)]$ are more complicated, but if $m = 1$ the standardized third and forth moments have the simple forms:

$$\gamma_1[R(t)] = \frac{\mathbb{E}[R(t)^3|S(0)]}{(Var[R(t)|S(0)])^{3/2}} = \frac{3\sigma}{2\sqrt{S(0)}}\left(\sqrt{\lambda t} + \frac{1}{\sqrt{\lambda t}}\right)$$

$$\gamma_2[R(t)] = \frac{\mathbb{E}[R(t)^4|S(0)]}{(\mathbb{E}[R(t)^2|S(0)])^2} - 3 = \frac{3}{\lambda t} + \frac{1}{S(0)\lambda t} + \frac{2\sigma^2}{S(0)}\left(\lambda t + 1 + \frac{1}{\lambda t}\right).$$

The estimates of m with daily frequency stock data are greater than but very close to one, so that the last two expressions give a close approximation to the higher moments over short trading periods. Skewness is always positive, decreasing in $S(0)$ and generally increasing with the period of the return t. Kurtosis is always positive, which means that it has fatter tails relative to the normal distribution, and it is a decreasing function of t for t less than $\sqrt{3S(0) + 2\sigma^2 + 1}/(\lambda\sigma\sqrt{2})$. The last expression is in general greater than 22, which means that the daily returns have fatter tails than the weekly and monthly returns. This characteristic has been observed for equity prices and is called aggregational normality (see Cont and Tankov [9]).

2.3 Option Pricing by BPRE

The discreteness of $S(t)$ under BPRE dynamics makes it impossible to replicate nonlinear payoff structures with only the underlying asset and riskless bonds. Accordingly, there is not a unique EMM within which derivatives can be priced as discounted expected values. Recall that the process $Z_n m^{-n}, n = 0, 1, 2, \ldots$ is a martingale (see, for example, Athreya and Ney [1]). The process $S(t)$ has a similar property, enabling us to find the EMM required for the option pricing.

Theorem 1. *Under conditions (i) and (ii) assumed in Section 2.2, the process* $S(t)e^{-\lambda t(m-1)}$, $t \geq 0$ *is a nonnegative martingale.*

The proof is as following. Since $S(t), t \geq 0$ is a continuous-time Markov chain, we have that for every n and every sequence $0 \leq t_1 < t_2 < \ldots < t_n < t$

$$\mathbb{E}[S(t)|S(t_n), \ldots, S(t_2), S(t_1)] = \mathbb{E}[S(t)|S(t_n)].$$

Therefore, it is sufficient to prove that for $t \geq 0$ and $\tau \geq 0$

$$\mathbb{E}[e^{-\lambda(t+\tau)(m-1)}S(t+\tau)|e^{-\lambda t(m-1)}S(t)] = e^{-\lambda t(m-1)}S(t). \tag{9}$$

Let us note first that

$$\mathbb{E}[e^{-\lambda(t+\tau)(m-1)}S(t+\tau)|e^{-\lambda t(m-1)}S(t)] = e^{-\lambda(t+\tau)(m-1)}\mathbb{E}[S(t+\tau)|S(t)]. \tag{10}$$

Using the fact that the processes $N(t)$ and Z_n are time-homogeneous, and using the main property of branching processes, we have

$$\begin{aligned} & \mathbb{E}[S(t+\tau)|S(t)] = \mathbb{E}[Z_{N(t+\tau)}|Z_{N(t)}] \\ = \ & \mathbb{E}[\sum_{i=1}^{Z_{N(t)}} Z^i_{N(t+\tau)-N(t)}|Z_{N(t)}] \\ = \ & \mathbb{E}[\sum_{i=1}^{Z_{N(t)}} Z^i_{N(\tau)}|Z_{N(t)}] = Z_{N(t)}\mathbb{E}[Z^1_{N(\tau)}] = S(t)\mathbb{E}[Z^1_{N(\tau)}], \end{aligned}$$

where $Z^i_{N(\tau)}$ are independent and identically distributed branching processes that are independent of $Z_{N(t)}$, and each of them starting with one particle. Using (8), with $S(0) = 1$, we get

$$\begin{aligned} & \mathbb{E}[S(t+\tau)|S(t)] = S(t)\mathbb{E}[Z_{N(\tau)}|Z_0 = 1] \\ = \ & S(t)\mathbb{E}[S(\tau)|S(0) = 1] = S(t)e^{\lambda\tau(m-1)}. \end{aligned} \tag{11}$$

Combining (9), (10), and (11) we complete the proof.

From (8) it follows that the discounted stock price process $S(t)e^{-rt}, t \in [0, T]$ has mean

$$\mathbb{E}[S(t)e^{-rt}|S(0)] = e^{[\lambda(m-1)-r]t}S(0).$$

Using Theorem 1, we can state that discounted stock price process $S(t)e^{-rt}$ will be a martingale if the parameters of the distribution of $S(t)$ are such that

$$\lambda(m-1) = r \Leftrightarrow \lambda\frac{a-p}{p} = r \Leftrightarrow a = p(1 + r/\lambda). \tag{12}$$

Utilizing the last relation we define EMM $\mathbb{Q}$ as follows:

1. We define $\mathbb{Q}$ to be identically equal to the real measure $\mathbb{P}$ on the elementary sets of the Poisson process, i.e. on sets of the form $\{N_{t_0} = n_0, N_{t_1} = n_1, \ldots, N_{t_k} = n_k\}$.

2. We define $\mathbb{Q}$ on the elementary sets of the branching process by

$$\mathbb{Q}(Z_{n+1} = 0|Z_n = 1) = (1 - \hat{a}),$$
$$\mathbb{Q}(Z_{n+1} = k|Z_n = 1) = \hat{a}p(1-p)^{k-1}, k = 1, 2, \ldots,$$
$$0 < \hat{a} < 1, \quad 0 < p < 1,$$

where $\hat{a}$ satisfies equation (12). In the case when $p(1+r/\lambda) \geq 1$, we also have to change the parameter p, $0 < p < 1$ in order to keep $\hat{a} \in (0, 1)$.

The choice of $\hat{a}$ guarantees that $0 < \hat{a} < 1$ and, therefore, we do not change the zero measure sets; that is, all sets that have zero measure with respect to the real measure $\mathbb{P}$ have zero measure with respect to $\mathbb{Q}$. Consequently, these two measures are equivalent. From the definition of $\mathbb{Q}$, it is easily seen that the discounted process $S(t)e^{-rt}$ is a martingale under $\mathbb{Q}$. Henceforth, we will work exclusively with the risk-neutral probability and will denote it simply by $\mathbb{P}$. In the numerical examples presented in the Section 4 we keep the estimated values of p and λ and choose the value of the parameter a so that equation (12) is satisfied.

Following Mitov and Mitov [21], we derive the formula for the price of a European call option

$$\begin{aligned}
C(0) &= e^{-rT}\mathbb{E}\left[\max\{S(T) - K, 0\}|S(0)\right] \\
&= e^{-rT}\left[\sum_{k=K+1}^{\infty} k\mathbb{P}(S(T) = k|S(0)) - K\mathbb{P}(S(T) > K|S(0))\right] \\
&= e^{-rT}\left[\mathbb{E}[S(T)|S(0)] - \sum_{k=1}^{K} k\mathbb{P}(S(T) = k|S(0)) - K(1 - \mathbb{P}(S(T) \leq K|S(0)))\right] \\
&= e^{-rT}\left[\mathbb{E}[S(T)|S(0)] - K\right] + e^{-rT}\sum_{k=1}^{K}(K-k)\mathbb{P}(S(T) = k|S(0)),
\end{aligned}$$

where the expectations are taken with respect to the risk-neutral measure $\mathbb{P}$ (the risk-neutral probability). Using the relation

$$\mathbb{P}(S(T) = k|S(0)) = \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} e^{-\lambda T} \mathbb{P}(Z_n = k|Z_0 = S(0)), k = 0, 1, 2, \ldots,$$

the fact that $S(t)e^{-\lambda t(m-1)}$ is a nonnegative martingale and (12), we obtain the following exact formula for the price of a European call option

$$C(0) = S(0) - e^{-rT}K + e^{-(r+\lambda)T} \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} \sum_{k=0}^{K} (K-k)\mathbb{P}(Z_n = k|Z_0 = S(0)), \quad (13)$$

where K is the strike price, T is time to maturity of the option, r is risk-free interest rate, $S(0)$ is the current stock price, and λ is the intensity of the Poisson process.

For practical purposes Mitov and Mitov use the approximation

$$C(0) \approx S(0) - Ke^{-rT} + e^{-(r+\lambda)T} \sum_{n=0}^{N} \frac{(\lambda T)^n}{n!} \sum_{k=0}^{K} (K-k)\mathbb{P}(Z_n = k|Z_0 = S(0)), \quad (14)$$

where number N can be determined in such a way that the error from the approximation will be less than ε. The probabilities $\mathbb{P}(Z_n = k|Z_0 = S(0))$ for $k = 0, 1, 2, \ldots K$ and $n = 1, 2, \ldots N$, are calculated by the iterative procedure described at the end of Section 2.1.

3 Barrier Option Pricing

There are several types of barrier options. Some "knock out" (i.e., they become worthless) when the underlying asset price crosses a barrier . If the underlying asset price begins below the barrier and must cross above it to cause the knock-out, the option is said to be up-and-out. A down-and-out option has the barrier below the initial asset price and knocks out if the asset price falls below the barrier. Other options "knock in" at a barrier (i.e., there is no payoff unless asset price crosses a barrier). Knock-in options also fall into two classes, up-and-in and down-and-in. The payoff at expiration for barrier options is typically either that of a put or a call. There exist more complex barrier options, but in this section, we focus only on an up-and-out call option on a BPRE process. The methodology we develop can be also applied to up-and-in call options. For the rest we can use in-out parity[1] and the price of the standard call option given in (13) and (14).

The price at maturity date of an up-and-out European call option with strike price K and barrier level B is:

$$C^{uo}(T) \stackrel{def}{=} \begin{cases} \max(0, S(T-K)), & M(T) \leq B, \\ 0, & M(T) > B \end{cases}$$

where $M(T) = \max_{0 \leq t \leq T} S(t)$. Applying martingale pricing, the price of the option at the current moment is

$$C^{uo}(0) = e^{-rT}\mathbb{E}[C^{uo}(T)|S(0)],$$

Recall that the expectation is taken with respect to the risk-neutral measure $\mathbb{P}$. Therefore

$$C^{uo}(0) = e^{-rT}\mathbb{E}[\max(0, S(T) - K)I_{\{M(T)\leq B\}}]$$

$$= e^{-rT}\sum_{j=0}^{B}\max(0, j-K)\mathbb{P}(S(T) = j, M(T) \leq B|S(0)).$$

Taking into account that $B > K$, we obtain[2]

$$C^{uo}(0) = e^{-rT}\sum_{j=K}^{B}(j-K)\mathbb{P}(S(T) = j, M(T) \leq B|S(0)).$$

Using the relation

$$\mathbb{P}(S(T) = k, M(T) \leq B|S(0)) = \sum_{n=0}^{\infty}\frac{(\lambda T)^n}{n!}e^{-\lambda T}\mathbb{P}(Z_n = k, M_n \leq B|Z_0 = S(0)),$$

for $k = 0, 1, 2, \ldots$, we can write

$$C^{uo}(0) = e^{-(r+\lambda)T}\sum_{n=0}^{\infty}\frac{(\lambda T)^n}{n!}\sum_{j=K}^{B}(j-K)\mathbb{P}(Z_n = j, M_n \leq B|Z_0 = S(0))$$

$$= e^{-(r+\lambda)T}\sum_{n=0}^{N}\frac{(\lambda T)^n}{n!}\sum_{j=K}^{B}(j-K)\mathbb{P}(Z_n = j, M_n \leq B|Z_0 = S(0))$$

[1]If we combine one "in" option and one "out" barrier option with the same strikes and expirations, we get the price of a standard option. This is the barrier option's equivalence to the put-call relationship for standard options.

[2]If barrier level B is less than or equal to the strike price K, the price of an up-and-out call option is zero. Therefore, we examine only the case where $B > K$.

$$+e^{-(r+\lambda)T}\sum_{n=N+1}^{\infty}\frac{(\lambda T)^n}{n!}\sum_{j=K}^{B}(j-K)\mathbb{P}(Z_n=j,M_n\leq B|Z_0=S(0)).$$

But since

$$\sum_{j=K}^{B}(j-K)\mathbb{P}(Z_n=j,M_n\leq B|Z_0)\leq\sum_{j=K}^{B}(j-K)1=\frac{(B-K)(B-K+1)}{2}$$

then

$$e^{-(r+\lambda)T}\sum_{n=N+1}^{\infty}\frac{(\lambda T)^n}{n!}\sum_{j=K}^{B}(j-K)\mathbb{P}(Z_n=j,M_n\leq B|Z_0=S(0))$$
$$\leq e^{-(r+\lambda)T}\sum_{n=N+1}^{\infty}\frac{(\lambda T)^n}{n!}\frac{(B-K)(B-K+1)}{2}.$$

Infinite series can be calculated with an arbitrary precision $\varepsilon>0$, hence the number N can be determined in such a way that

$$e^{-(r+\lambda)T}\frac{(B-K)(B-K+1)}{2}\sum_{n=N+1}^{\infty}\frac{(\lambda T)^n}{n!}<\varepsilon,$$

provided the values of r, λ, T, B, K, and $S(0)$ are known. Therefore, we can use the approximation

$$C^{uo}(0)\approx e^{-(r+\lambda)T}\sum_{n=0}^{N}\frac{(\lambda T)^n}{n!}\sum_{j=K}^{B}(j-K)\mathbb{P}(Z_n=j,M_n\leq B|Z_0=S(0)),\tag{15}$$

with an error less than ε.

The method for the calculation of the probabilities $\mathbb{P}(Z_n=j,M_n\leq B|Z_0=S(0))$ used in equation (15), is given in the following theorem.

Theorem 2. *If $\{Z_n,n=0,1,2,\ldots\}$ is a Bienayme-Galton-Watson branching process with $p_{ij}=\mathbb{P}(Z_1=j|Z_0=i),i,j=0,1,2,\ldots$ and B is a positive integer then the probability*

$$\mathbb{P}(Z_n=j,M_n\leq B|Z_0=i)$$

is given by the $(i,j)-$th element of the $n-$th power of matrix $\mathbf{P}(B,1)$, where

$$p_{ij}(B,1)=\mathbb{P}(Z_1=j,M_1\leq B|Z_0=i)=\begin{cases}p_{ij}, & 1\leq i,j\leq B,\\ 1, & i,j=0,\\ 0, & i=0,j>0,\\ 0, & B<i,j\end{cases}$$

The proof of this theorem is as follows. For the conditional probabilities $\mathbb{P}(Z_2 = j, M_2 \leq B | Z_0 = i)$, calculated for the second generation, we can write:

$$
\begin{aligned}
p_{ij}(B,2) &= \mathbb{P}(Z_2 = j, M_2 \leq B | Z_0 = i) \\
&= \sum_{l=0}^{\infty} \mathbb{P}(Z_2 = j, M_2 \leq B | Z_1 = l)\mathbb{P}(Z_1 = l, M_1 \leq B | Z_0 = i) \\
&= \sum_{l=1}^{B} \mathbb{P}(Z_2 = j, M_2 \leq B | Z_1 = l)\mathbb{P}(Z_1 = l, M_1 \leq B | Z_0 = i) \\
&= \sum_{l=1}^{B} \mathbb{P}(Z_1 = j, M_1 \leq B | Z_0 = l)\mathbb{P}(Z_1 = l, M_1 \leq B | Z_0 = i) \\
&= \sum_{l=1}^{B} p_{il}p_{lj}, \quad 1 \leq i, j \leq B.
\end{aligned}
$$

Therefore $\mathbf{P}(B,2) = (p_{ij}(B,2))_{i,j=1,\ldots,B} = \mathbf{P}(B,1)^2$. Repeating the above procedure, we can deduce by induction that $\mathbf{P}(B,n) = \mathbf{P}(B,1)^n$, which proves the statement of the theorem.

The one-step transition probabilities $p_{ij} = \mathbb{P}(Z_1 = j | Z_0 = i), \quad i, j = 0, 1, 2, \ldots$, are calculated using the iterative procedure described at the end of Section 2.1.

4 Numerical Examples

In this section, we compare the prices of up-and-out barrier options calculated under the lognormal and BPRE models. According to Musiela and Rutkowski [22], any mathematical model used for pricing exotic options should be marked-to-market; that is, at any given date it should reproduce with the desired precision the current market prices of liquid options. Standard options are the most liquid options and therefore we mark-to-market our model by minimizing the sum-of-squares distance between the theoretical option values and market prices of standard options. The estimated parameters are then fed into the lognormal and BPRE models to calculate the up-and-out barrier option prices.

4.1 Parameters Estimation

We estimate the EMM parameters of the BPRE and lognormal model for the S&P 500 Index (SPX) and the following three stocks: Intel (INTC), Microsoft (MSFT), and

Amazon.com (AMZN). The data were obtained from Option Metrics's IvyDB in the Wharton Research Data Services. We selected five mid-month Wednesdays (June 8, 2005; July 13, 2005; August 10, 2005; September 14, 2005; October 12, 2005) and estimate the EMM parameters of the stock price processes. The market option prices are computed by using the Black-Scholes formula with the implied volatilities and dividends given by IvyDB. For the daily risk-free rate, we select the appropriate zero-coupon rate supplied by IvyDB and convert it to a continuous-compound rate.

For each model, we estimate the EMM parameters using the method of least-squares calibration with a prior (see Kim et al. [16]); that is, we estimate them by nonlinear least squares minimization under the EMM condition (12). The BPRE model has three EMM parameters to be estimated and one restriction in its EMM condition, while the lognormal model has one EMM parameter and there are no additional restrictions. Hence, the BPRE model has two free parameter (λ and p), while the lognormal model has only one free parameter (σ) for the estimation.

To measure the performance of the prices estimated from the two models, we use average absolute error (AAE), average percentage error (APE), and root-mean square error (RMSE). These are defined as follows (Schoutens [28]):

$$AAE = \sum_{\text{options}} \frac{|\text{market price} - \text{model price}|}{\text{number of options}},$$

$$APE = \frac{1}{\text{mean option price}} \sum_{\text{options}} \frac{|\text{market price} - \text{model price}|}{\text{number of options}},$$

$$RMSE = \sqrt{\sum_{\text{options}} \frac{(\text{market price} - \text{model price})^2}{\text{number of options}}}.$$

The values of these errors and estimated parameters for both models are presented in Tables 1 through 4.

4.2 Up-and-Out Call Option Prices

Using the estimated parameters, we calculate the prices of up-and-out call options with different maturities and strikes, while the barrier level is the same for all cases. It is chosen to be greater than all of the strikes. The results for SPX, AMZN, INTC, and MSFT are reported in Tables 5 through 8. In the same tables, we report the results for standard call options with the same strikes and maturities. The BPRE prices of barrier and standard options are calculated by (15) and (14), respectively. The prices of up-and-out call options for the BPRE process deviate significantly from those for the lognormal process, while the prices of the corresponding standard options are similar. For example, if we examine the options on SPX with 72 days to maturity (Table 5), we can see that the maximum percentage differences in barrier option prices is 139.53%, while for the standard options it is 1.15%.

It can be seen that the BPRE prices are less than the lognormal prices for the up-and-out options with shorter maturity (five days). This is due to the fact that BPRE returns have fatter tails for smaller periods and therefore we have greater probability of reaching the barrier level with respect to the lognormal model. In other words, if we have the same value for the standard call option, we will get the smaller value for an up-and-out barrier option using BPRE when the maturity period is shorter. It is interesting to note that the BPRE model produces greater prices for the options with 38 and 72 days to expiration, compared to the lognormal model. Figures 1 through 4 show the barrier option prices as a function of time to maturity. This could be partially explained by the presence of the so called "aggregational normality" effect, i.e. the distribution of the return gets closer to the normal distribution as the length of its period increases. Therefore, the difference in the tails of the distribution has a significant impact on barrier option prices.

5 Conclusions

In this paper, we present a simple and easy-to-use method for computing accurate estimates of up-and-out call option prices when the underlying stock process is modeled by BPRE. We demonstrate that the prices of barrier options for the BPRE process can deviate significantly from those calculated assuming a lognormal process, even if we have similar values for the corresponding standard options. We find that there is different behavior for the prices computed from the BPRE model with respect to the maturity period. For shorter maturities, the BPRE model gives smaller values compared to the lognormal model, while for the longer times to expiration we observe the opposite.

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Table 1: SPX : Estimated EMM parameters and respective estimation errors

Date	Model	Parameters			AAE	APE	RMSE
08-JUN-2005	BPRE	p	a	λ			
		0.9891	0.9892	1.3968	1.1457	0.0120	1.3836
	lognormal	σ					
		0.0827			1.0391	0.0109	1.3387
13-JUL-2005	BPRE	p	a	λ			
		0.9913	0.9914	1.3202	1.3723	0.0129	1.845
	lognormal	σ					
		0.2009			1.3375	0.0125	1.8392
10-AUG-2005	BPRE	p	a	λ			
		0.9885	0.9886	1.4149	0.7787	0.0085	0.9559
	lognormal	σ					
		0.0822			0.7947	0.0087	0.9638
14-SEP-2005	BPRE	p	a	λ			
		0.9883	0.9884	1.4197	1.1849	0.0108	1.4335
	lognormal	σ					
		0.0831			1.1807	0.0107	1.4324
12-OCT-2005	BPRE	p	a	λ			
		0.9887	0.9888	1.6182	0.6793	0.0473	0.9077
	lognormal	σ					
		0.0917			0.4178	0.0295	0.7989

Table 2: AMZN : Estimated EMM parameters and respective estimation errors

Date	Model	Parameters			AAE	APE	RMSE
08-JUN-2005	BPRE	p	a	λ			
		0.9786	0.9787	1.6634	0.0814	0.0325	0.0872
	lognormal	σ					
		0.2635			0.0845	0.0338	0.0903
13-JUL-2005	BPRE	p	a	λ			
		0.9650	0.9651	1.9035	0.1195	0.0281	0.1269
	lognormal	σ					
		0.3445			0.1441	0.0340	0.1514
10-AUG-2005	BPRE	p	a	λ			
		0.9143	0.9145	0.6477	0.0435	0.0131	0.0540
	lognormal	σ					
		0.2925			0.0541	0.0162	0.0675
14-SEP-2005	BPRE	p	a	λ			
		0.9663	0.9664	1.8833	0.0391	0.0115	0.0481
	lognormal	σ					
		0.3177			0.0463	0.0137	0.0585
12-OCT-2005	BPRE	p	a	λ			
		0.9387	0.9388	2.3054	0.0349	0.0074	0.0415
	lognormal	σ					
		0.4747			0.0771	0.0164	0.0885

Table 3: INTC : Estimated EMM parameters and respective estimation errors

Date	Model	Parameters			AAE	APE	RMSE
08-JUN-2005	BPRE	p	a	λ			
		0.9850	0.9851	1.5135	0.0329	0.0273	0.0361
	lognormal	σ					
		0.2456			0.0219	0.0153	0.0274
13-JUL-2005	BPRE	p	a	λ			
		0.9860	0.9861	1.4860	0.0452	0.0320	0.0548
	lognormal	σ					
		0.2308			0.0379	0.0269	0.0453
10-AUG-2005	BPRE	p	a	λ			
		0.9891	0.9892	1.3954	0.0499	0.0364	0.0516
	lognormal	σ					
		0.1895			0.0557	0.0407	0.0579
14-SEP-2005	BPRE	p	a	λ			
		0.9859	0.9860	1.4893	0.0471	0.0425	0.0552
	lognormal	σ					
		0.2535			0.0564	0.0512	0.0577
12-OCT-2005	BPRE	p	a	λ			
		0.9813	0.9814	1.6031	0.0492	0.0344	0.0612
	lognormal	σ					
		0.3091			0.0702	0.0493	0.0759

Table 4: MSFT : Estimated EMM parameters and respective estimation errors

Date	Model	Parameters			AAE	APE	RMSE
08-JUN-2005	BPRE	p	a	λ			
		0.9227	0.9235	0.1413	0.0670	0.0489	0.0791
	lognormal	σ					
		0.1697			0.0748	0.0550	0.0862
13-JUL-2005	BPRE	p	a	λ			
		0.9874	0.9875	1.2226	0.0350	0.0289	0.0355
	lognormal	σ					
		0.2009			0.0446	0.0371	0.0463
10-AUG-2005	BPRE	p	a	λ			
		0.9375	0.9382	0.1898	0.0384	0.1568	0.0424
	lognormal	σ					
		0.1819			0.0591	0.0606	0.0730
14-SEP-2005	BPRE	p	a	λ			
		0.9913	0.9915	1.3185	0.0212	0.0216	0.0213
	lognormal	σ					
		0.1740			0.0143	0.0146	0.0144
12-OCT-2005	BPRE	p	a	λ			
		0.9103	0.9110	0.1945	0.0265	0.0278	0.0316
	lognormal	σ					
		0.2298			0.0409	0.0430	0.0506

Table 5: SPX : Up-and-Out and Standard Call Option Prices

Up-and-Out Call Option Prices

	5 days to Maturity		38 days to Maturity		72 days to Maturity	
Strike	BPRE	lognormal	BPRE	lognormal	BPRE	lognormal
1140	87.8365	88.0095	73.9546	70.2746	53.3410	49.0356
1160	67.8538	68.0250	56.9195	53.7793	40.3445	36.7079
1180	47.8757	48.0417	40.6091	38.0147	28.3509	25.4005
1200	28.0554	28.1845	25.9172	23.8965	17.9533	15.7139
1220	10.4329	10.6313	14.0213	12.5981	9.7494	8.2266
1240	1.6013	1.6579	5.8552	5.0065	4.1346	3.2765
1260	0.0979	0.0719	1.5141	1.1501	1.0921	0.7582
1270	0.0018	0.0005	0.0962	0.0444	0.0706	0.0295

Standard Call Option Prices

	5 days to Maturity		38 days to Maturity		72 days to Maturity	
Strike	BPRE	lognormal	BPRE	lognormal	BPRE	lognormal
1140	88.0133	88.0133	93.7358	93.7159	100.0277	99.9622
1160	68.0284	68.0283	74.1869	74.1452	81.2017	81.1114
1180	48.0482	48.0445	55.3628	55.3053	63.3820	63.2865
1200	28.2256	28.1868	38.1574	38.1121	47.1707	47.1026
1220	10.6010	10.6330	23.7500	23.7426	33.1932	33.1817
1240	1.7564	1.6591	13.0843	13.1021	21.9192	21.9705
1260	0.2344	0.0727	6.3007	6.2990	13.5034	13.5943
1270	0.1196	0.0007	2.6617	2.6066	7.7357	7.8258

The current value $S(0)$ of the SPX is 1227.2 and the risk-free rate r is 0.0377 per annum. The barrier level is 1290 in order to be greater than all of the selected strike prices.

Table 6: AMZN : Up-and-Out and Standard Call Option Prices

Up-and-Out Call Option Prices						
	5 days to Maturity		38 days to Maturity		72 days to Maturity	
Strike	BPRE	lognormal	BPRE	lognormal	BPRE	lognormal
39	4.0173	4.0962	2.3249	1.9097	1.3452	1.0368
40	3.0510	3.1285	1.8005	1.4481	1.0270	0.7718
41	2.1410	2.2189	1.3410	1.0517	0.7551	0.5511
42	1.3476	1.4268	0.9519	0.7240	0.5301	0.3738
43	0.7394	0.8116	0.6360	0.4656	0.3507	0.2373
44	0.3475	0.3989	0.3923	0.2735	0.2146	0.1379
45	0.1388	0.1649	0.2167	0.1412	0.1178	0.0706
46	0.0462	0.0548	0.1013	0.0597	0.0548	0.0297
Standard Call Option Prices						
	5 days to Maturity		38 days to Maturity		72 days to Maturity	
Strike	BPRE	lognormal	BPRE	lognormal	BPRE	lognormal
39	4.1412	4.1369	4.8860	4.8689	5.6102	5.5882
40	3.1726	3.1652	4.1298	4.1182	4.9179	4.9069
41	2.2603	2.2515	3.4395	3.4348	4.2772	4.2789
42	1.4646	1.4553	2.8213	2.8239	3.6905	3.7053
43	0.8541	0.8361	2.2790	2.2879	3.1589	3.1865
44	0.4599	0.4193	1.8132	1.8264	2.6825	2.7217
45	0.2488	0.1813	1.4221	1.4367	2.2603	2.3091
46	0.1539	0.0671	1.1010	1.1136	1.8904	1.9462

The current price $S(0)$ of the AMZN is 43.10 and the risk-free rate r is 0.0377 per annum. The barrier level is 49 in order to be greater than all of the selected strike prices.

Table 7: INTC : Up-and-Out and Standard Call Option Prices

Up-and-Out Call Option Prices						
	5 days to Maturity		38 days to Maturity		72 days to Maturity	
Strike	BPRE	lognormal	BPRE	lognormal	BPRE	lognormal
23.00	1.4069	1.5200	1.7159	1.6759	1.6180	1.3661
23.50	0.9455	1.0580	1.3699	1.3459	1.3267	1.1102
24.00	0.5490	0.6570	1.0652	1.0548	1.0662	0.8836
24.50	0.2641	0.3530	0.8047	0.8046	0.8378	0.6867
25.00	0.1045	0.1598	0.5889	0.5953	0.6414	0.5190
25.50	0.0346	0.0599	0.4160	0.4254	0.4765	0.3797
26.00	0.0099	0.0184	0.2823	0.2918	0.3415	0.2670
26.50	0.0025	0.0046	0.1829	0.1905	0.2342	0.1786
Standard Call Option Prices						
	5 days to Maturity		38 days to Maturity		72 days to Maturity	
Strike	BPRE	lognormal	BPRE	lognormal	BPRE	lognormal
23.00	1.5219	1.5200	1.9352	1.9557	2.2888	2.3274
23.50	1.0605	1.0580	1.5820	1.6058	1.9597	2.0041
24.00	0.6640	0.6570	1.2702	1.2948	1.6615	1.7102
24.50	0.3790	0.3530	1.0026	1.0247	1.3953	1.4463
25.00	0.2191	0.1598	0.7796	0.7956	1.1613	1.2119
25.50	0.1490	0.0599	0.5995	0.6059	0.9588	1.0064
26.00	0.1239	0.0184	0.4587	0.4525	0.7864	0.8283
26.50	0.1163	0.0046	0.3522	0.3315	0.6420	0.6756

The current price $S(0)$ of the INTC is 24.49 and the risk-free rate r is 0.0377 per annum. The barrier level is 30 in order to be greater than all of the selected strike prices.

Table 8: MSFT : Up-and-Out and Standard Call Option Prices

Up-and-Out Call Option Prices						
	5 days to Maturity		38 days to Maturity		72 days to Maturity	
Strike	BPRE	lognormal	BPRE	lognormal	BPRE	lognormal
25.00	1.2742	1.3326	1.5696	1.6149	1.7205	1.6518
25.50	0.8000	0.8576	1.1982	1.2469	1.3829	1.3280
26.00	0.3975	0.4534	0.8787	0.9292	1.0844	1.0425
26.50	0.1427	0.1819	0.6166	0.6661	0.8275	0.7968
27.00	0.0374	0.0518	0.4125	0.4579	0.6126	0.5910
27.50	0.0076	0.0100	0.2624	0.3010	0.4384	0.4235
28.00	0.0012	0.0013	0.1582	0.1883	0.3017	0.2915
28.50	0.0002	0.0001	0.0900	0.1115	0.1983	0.1909
Standard Call Option Prices						
	5 days to Maturity		38 days to Maturity		72 days to Maturity	
Strike	BPRE	lognormal	BPRE	lognormal	BPRE	lognormal
25.00	1.3342	1.3326	1.6395	1.6452	1.9252	1.9374
25.50	0.8600	0.8576	1.2674	1.2752	1.5777	1.5935
26.00	0.4575	0.4534	0.9472	0.9552	1.2693	1.2878
26.50	0.2024	0.1819	0.6844	0.6900	1.0025	1.0220
27.00	0.0964	0.0518	0.4797	0.4797	0.7777	0.7960
27.50	0.0658	0.0100	0.3289	0.3205	0.5935	0.6084
28.00	0.0588	0.0013	0.2240	0.2057	0.4470	0.4562
28.50	0.0571	0.0001	0.1551	0.1268	0.3337	0.3357

The current price $S(0)$ of the MSFT is 26.31 and the risk-free rate r is 0.0377 per annum. The barrier level is 32 in order to be greater than all of the selected strike prices.

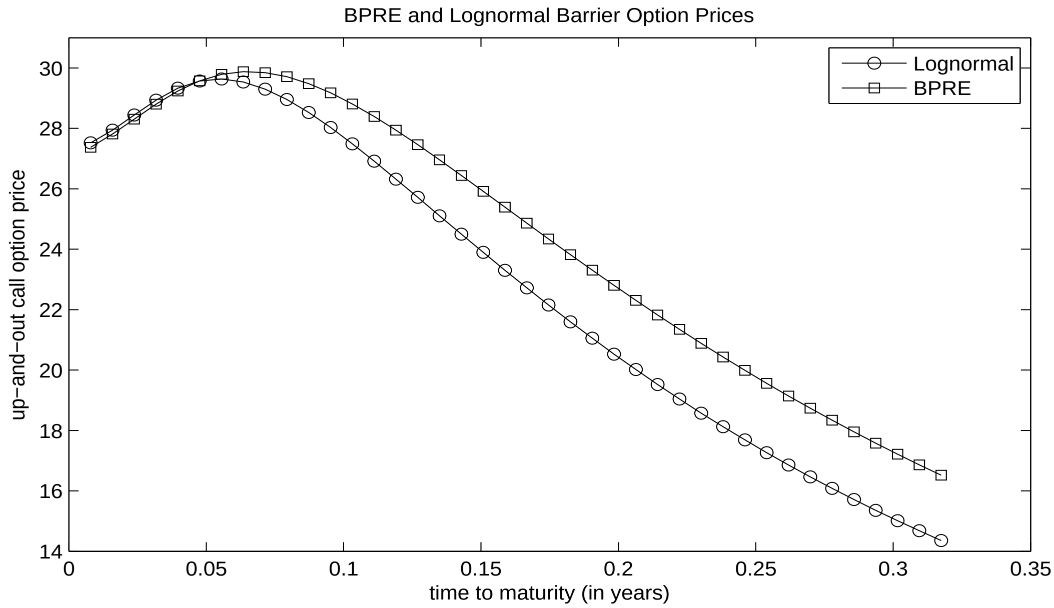

Figure 1: Up-and-out call option on SPX prices from BPRE model and lognormal model as a function of time to maturity. The current value $S(0)$ of the SPX is 1227.2, the risk-free rate r is 0.0377 per annum, the strike price is 1200, and the barrier level is 1290.

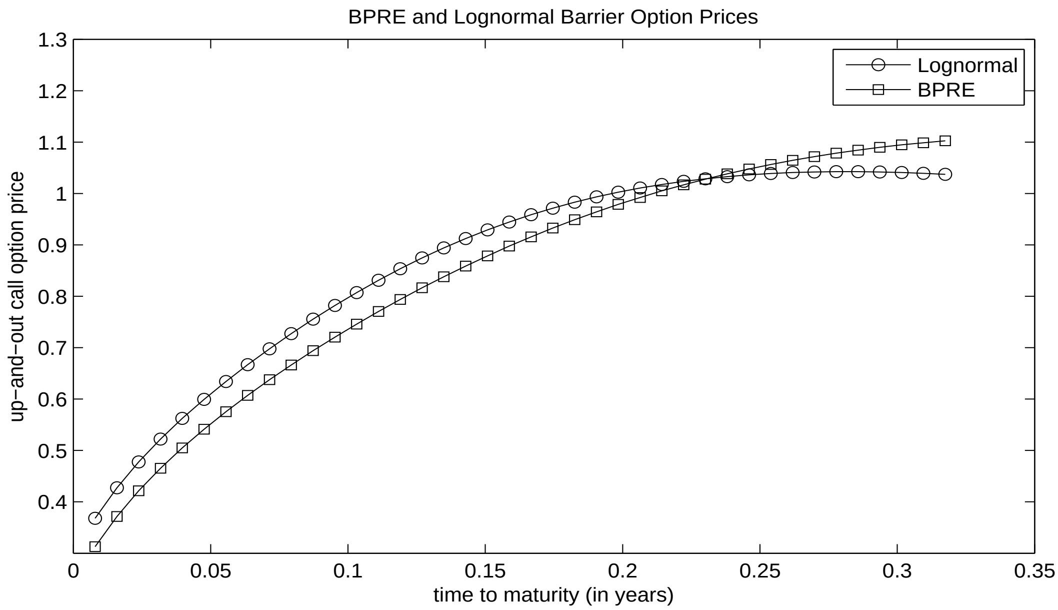

Figure 2: Up-and-out call option on AMZN prices from BPRE model and lognormal model as a function of time to maturity. The current price $S(0)$ of the AMZN is 43.1, the risk-free rate r is 0.0377 per annum, the strike price is 42, and the barrier level is 49.

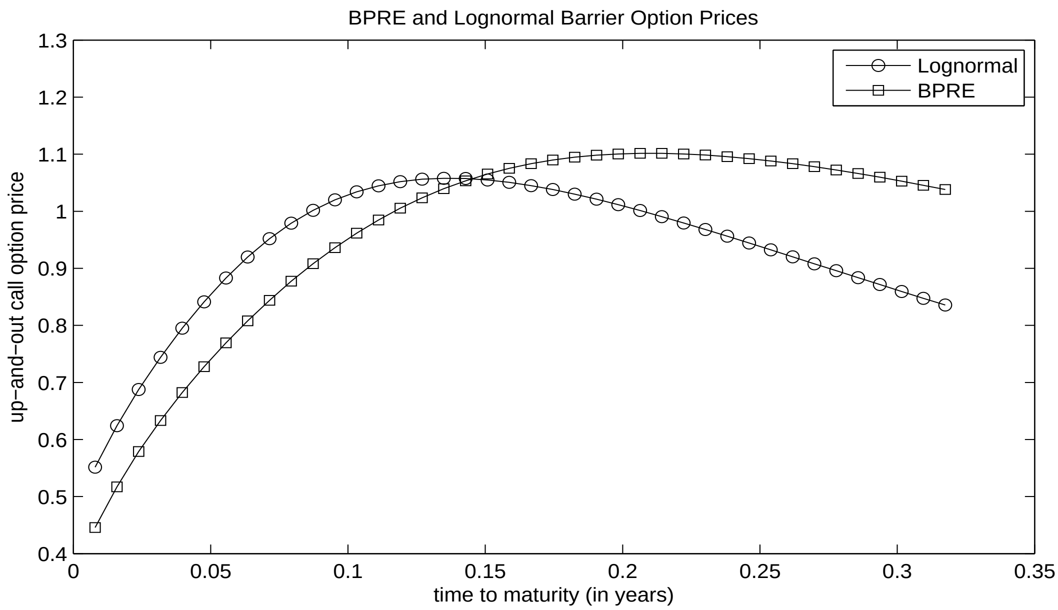

Figure 3: Up-and-out call option on INTC prices from BPRE model and lognormal model as a function of time to maturity. The current price $S(0)$ of the INTC is 24.49, the risk-free rate r is 0.0377 per annum, the strike price is 24, and the barrier level is 30.

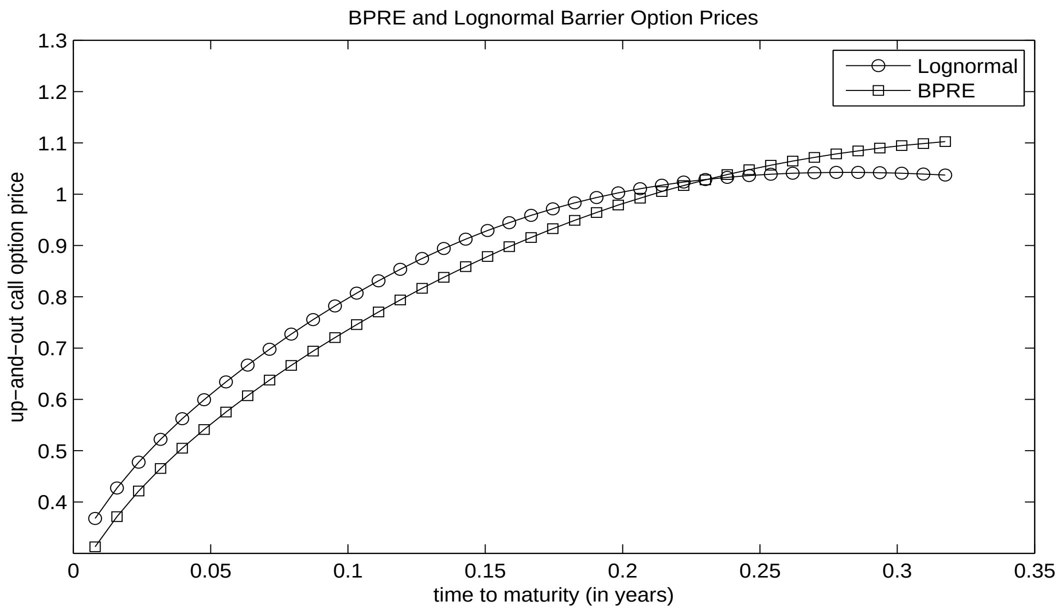

Figure 4: Up-and-out call option on MSFT prices from BPRE model and lognormal model as a functio,n of time to maturity. The current price $S(0)$ of the MSFT is 26.31, the risk-free rate r is 0.0377 per annum, the strike price is 26, and the barrier level is 32.